Leaky Capacitors

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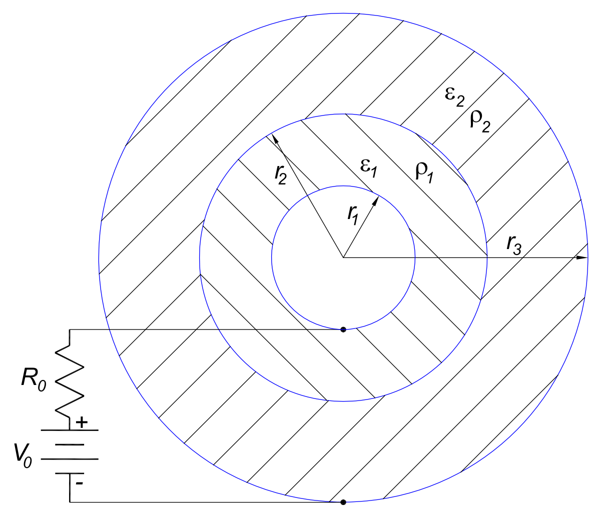
1 Problem

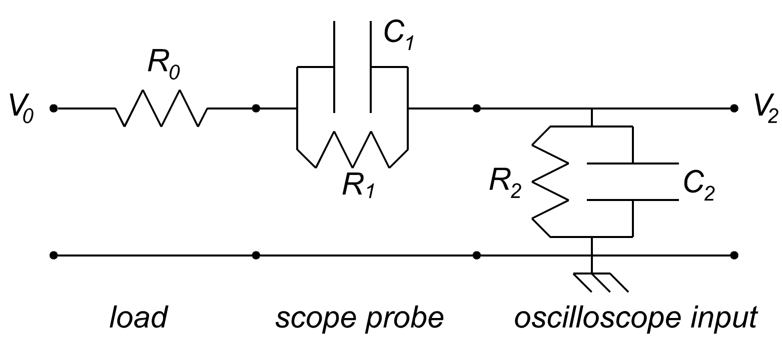
A battery of voltage V and internal resistance R_0 is connected across a pair of "leaky" spherical capacitors that have inner electrode of radius r_1, intermediate electrode of radius r_2, and outer electrode of radius r_3 and whose gaps are filled with concentric shells of dielectrics of differing relative permittivities, ϵ_1 for $r_1 < r < r_2$ and ϵ_2 for $r_2 < r < r_3$. The dielectrics are not perfect insulators but have resistivities ρ_1 and ρ_2, respectively.



Find the voltage $V_2(t)$ of electrode 2 supposing that the battery is connected at time $t = 0$.

The human body contains about 10^{16} *"leaky capacitors" = the synapses of your nervous system.*

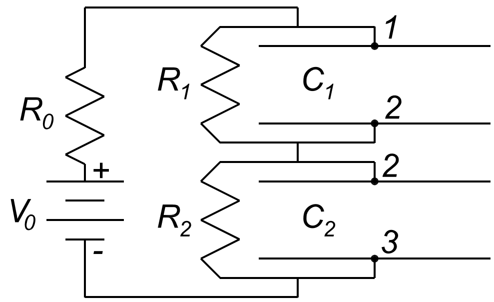
An application of a pair of "leaky capacitors" is shown below, in which the input circuit of an oscilloscope and a passive probe each include a capacitor and resistor in series.



2 Solution

2.1 Resistance and Capacitance

Each of the spherical shells is equivalent to a resistor in parallel with a capacitor, and the two shells are in series, so an equivalent circuit for this problem is as shown below.



The resistance R of a spherical shell of resistivity ρ, inner radius r_1 and outer radius r_2 is given by,

$$R = \int_{r_1}^{r_2} \frac{\rho\, dr}{4\pi r^2} = \frac{\rho(r_2 - r_1)}{4\pi r_1 r_2}. \tag{1}$$

To calculate the capacitance C of a spherical shell capacitor filled with a dielectric of relative permittivity ϵ, consider charge Q placed on the electrode at radius r_1, such that the radial electric field is $E = D/\epsilon = Q/4\pi\epsilon\, r^2$ (in Gaussian units), and,

$$V = \frac{Q}{C} = \int_{r_1}^{r_2} E\, dr = \int_{r_1}^{r_2} \frac{Q\, dr}{4\pi\epsilon\, r^2} = \frac{Q(r_2 - r_1)}{4\pi\epsilon\, r_1 r_2}, \tag{2}$$

so the capacitance is,

$$C = \frac{4\pi\epsilon\, r_1 r_2}{r_2 - r_1}. \tag{3}$$

Applying these results to the case of two shells, $r_1 < r < r_2$ and $r_2 < r < r_3$, with permittivities and resistivities $\epsilon_1,\ \rho_1$ and $\epsilon_2,\ \rho_2$, respectively, we have,

$$R_1 = \frac{\rho_1(r_2 - r_1)}{4\pi r_1 r_2}, \qquad R_2 = \frac{\rho_2(r_3 - r_2)}{4\pi r_2 r_3}, \qquad C_1 = \frac{4\pi\epsilon_1\, r_1 r_2}{r_2 - r_1}, \qquad C_2 = \frac{4\pi\epsilon_2\, r_2 r_3}{r_3 - r_2}. \tag{4}$$

2.2 One Leaky Capacitor

We first consider the case of a single shell, with resistance R and capacitance C, in series with an "external" resistor R_0 and a battery of voltage V.

At large times the current I is steady, with the value,

$$I_\infty = \frac{V}{R_0 + R}, \tag{5}$$

while at $t = 0$, when the battery is first connected, the current is,

$$I_0 = \frac{V}{R_0}, \tag{6}$$

since the capacitor appear as a short initially.

Within the shell, the current can be thought of as splitting into two parts that flow (in parallel) through the resistor and capacitor,

$$I(t) = I_R(t) + I_C(t), \tag{7}$$

where $I_R(0) = 0$, $I_R(\infty) = V_0/(R_0 + R)$, $I_C(0) = V_0/R_0$ and $I_C(\infty) = 0$.

The voltage across the shell is given by $I_R R = Q_C/C$, whose time derivative is $\dot{I}_R = I_C/RC$. Then, $I_C(t) = \dot{I}_R(t)RC$, and the total current (7) can be written as,

$$I = I_R + \dot{I}_R RC. \tag{8}$$

Kirchhoff's law for the loop that includes the resistance R is now,

$$V_0 = IR_0 + I_R R_1 = \dot{I}_R R_0 RC + I_R(R_0 + R), \tag{9}$$

which implies that the current in resistor R is,

$$I_R(t) = \frac{V_0}{R_0 + R}\left(1 - e^{-(R_0+R)t/R_0RC}\right). \tag{10}$$

Hence, the current in the capacitor is,

$$I_C(t) = RC\dot{I}_R = \frac{V_0}{R_0}e^{-(R_0+R)t/R_0RC}, \tag{11}$$

and the total current is,

$$I(t) = \frac{V_0}{R_0 + R}\left(1 - e^{-(R_0+R)t/R_0RC}\right) + \frac{V_0}{R_0}e^{-(R_0+R)t/R_0RC} = \frac{V_0}{R_0 + R}\left(1 + \frac{R}{R_0}e^{-(R_0+R)t/R_0RC}\right). \tag{12}$$

2.3 Two Leaky Capacitors

In the original problem with two shells, each with resistance R_i and capacitance C_i, $i = 1, 2$, the steady current at long times is,

$$I(\infty) = \frac{V_0}{R_0 + R_1 + R_2}. \tag{13}$$

The conseptual splitting of the current into parallel paths within each shell can now be written as,

$$I = I_{R_1} + I_{C_1} = I_{R_2} + I_{C_2}. \tag{14}$$

Extrapolating the form (10) to the case of two shells, we anticipate the presence of two time constants τ_i,

$$I_{R_i} = \frac{V_0}{R_0 + R_1 + R_2}\left(1 - \alpha_i e^{-t/\tau_\alpha} - \beta_i e^{-t/\tau_\beta}\right), \tag{15}$$

where,

$$\alpha_i + \beta_i = 1, \tag{16}$$

so that $I_{R_i}(\infty) = I(\infty) = V_0/(R_0 + R_1 + R_2)$. Both exponentials e^{-t/τ_α} and e^{-t/τ_β} appear in both currents I_{R_i} so that eq. (14) can hold at all times.

The currents in the capacitors are,

$$I_{C_i} = R_i C_i \dot{I}_{R_i} = \frac{V_0 R_i C_i}{R_0 + R_1 + R_2} \left(\frac{\alpha_i}{\tau_\alpha} e^{-t/\tau_\alpha} + \frac{\beta_i}{\tau_\beta} e^{-t/\tau_\beta} \right). \tag{17}$$

The initial condition on the currents in the capacitors are that $I_{C_i}(0) = V_0/R_0$, so we obtain two constraints,

$$\frac{\alpha_i}{\tau_\alpha} + \frac{\beta_i}{\tau_\beta} = \frac{R_0 + R_1 + R_2}{R_0 R_i C_i} \equiv \frac{1}{T_i}, \tag{18}$$

where,

$$T_i = \frac{R_0}{R_0 + R_1 + R_2} \tau_i, \qquad \text{and} \qquad \tau_i = R_i C_i. \tag{19}$$

We can express the α_i and β_i in terms of τ_α and τ_β using eq. (16) in (18),

$$\alpha_i = \frac{\tau_\alpha}{\tau_\beta - \tau_\alpha} \frac{\tau_\beta - T_i}{T_i}, \qquad \text{and} \qquad \beta_i = \frac{\tau_\beta}{\tau_\alpha - \tau_\beta} \frac{\tau_\alpha - T_i}{T_i}. \tag{20}$$

Two more constraints are needed to deduce τ_α and τ_β. One of these can be obtained by using eqs. (15) and (17) in (14),

$$\alpha_1 \left(\frac{\tau_1}{\tau_\alpha} - 1 \right) e^{-t/\tau_\alpha} + \beta_1 \left(\frac{\tau_1}{\tau_\beta} - 1 \right) e^{-t/\tau_\beta} = \alpha_2 \left(\frac{\tau_2}{\tau_\alpha} - 1 \right) e^{-t/\tau_\alpha} + \beta_2 \left(\frac{\tau_2}{\tau_\beta} - 1 \right) e^{-t/\tau_\beta}. \tag{21}$$

For this to hold at all times, we must have,

$$\alpha_1 (\tau_1 - \tau_\alpha) = \alpha_2 (\tau_2 - \tau_\alpha) \qquad \text{and} \qquad \beta_1 (\tau_1 - \tau_\beta) = \beta_2 (\tau_2 - \tau_\beta). \tag{22}$$

Using this in eq. (20) we find,

$$T_2(\tau_1 - \tau_\alpha)(\tau_\beta - T_1) = T_1(\tau_2 - \tau_\alpha)(\tau_\beta - T_2), \tag{23}$$

and,

$$T_2(\tau_1 - \tau_\beta)(\tau_\alpha - T_1) = T_1(\tau_2 - \tau_\beta)(\tau_\alpha - T_2). \tag{24}$$

From either eq. (23) or (24) we have,

$$\tau_\beta = \frac{T_1 T_2 (\tau_1 - \tau_2)}{\tau_1 T_2 - \tau_2 T_1 + \tau_\alpha (T_1 - T_2)} = \frac{T_1 T_2 (\tau_1 - \tau_2)}{\tau_\alpha (T_1 - T_2)} = \frac{R_0 \tau_1 \tau_2}{\tau_\alpha (R_0 + R_1 + R_2)}, \tag{25}$$

recalling eq. (19).

The final constraint can be obtained from Kirchhoff's law for the loop containing resistors R_0, R_1 and R_2,

$$\begin{aligned} V_0 &= I R_0 + I_{R_1} R_1 + I_{R_2} R_2 \\ &= I_{R_1}(R_0 + R_1) + I_{C_1} R_0 + I_{R_2} R_2 \\ &= V_0 + \frac{V_0 \, e^{-t/\tau_\alpha}}{(R_0 + R_1 + R_2)\tau_\alpha} [\alpha_1 (R_0 R_1 C_1 - (R_0 + R_1)\tau_\alpha) - \alpha_2 R_2 \tau_\alpha] \\ &\qquad + \frac{V_0 \, e^{-t/\tau_\beta}}{(R_0 + R_1 + R_2)\tau_\beta} [\beta_1 (R_0 R_1 C_1 - (R_0 + R_1)\tau_\beta) - \beta_2 R_2 \tau_\beta]. \end{aligned} \tag{26}$$

For this to be true at all times, each of the quantities in brackets must vanish. Using, for example, the second bracket together with eqs. (20) and (25) we find (after dividing out the common factor $R_0R_1C_1$) a quadratic equation for τ_α,

$$\tau_\alpha^2 - A\tau_\alpha + B = 0, \tag{27}$$

where,

$$A = \frac{(R_0 + R_2)R_1C_1 + (R_0 + R_1)R_2C_2}{R_0 + R_1 + R_2} = \frac{R_0(\tau_1 + \tau_2) + R_1\tau_2 + R_2\tau_1}{R_0 + R_1 + R_2}, \tag{28}$$

and,

$$B = \frac{R_0R_1C_1R_2C_2}{R_0 + R_1 + R_2} = \frac{R_0\tau_1\tau_2}{R_0 + R_1 + R_2}. \tag{29}$$

The quadratic equation (27) has two positive roots,

$$\tau_\alpha = \frac{A + \sqrt{A^2 - 4B}}{2}, \qquad \text{and} \qquad \tau_\beta = \frac{A - \sqrt{A^2 - 4B}}{2} = \frac{B}{\tau_\alpha}, \tag{30}$$

recalling eq. (25). As a check, note that if $R_2 = C_2 = 0$ then $B = 0$ and $\tau_\alpha = A = R_0R_1C_1/(R_0 + R_1)$, as found in eq. (11).

We note that $\tau_\alpha\tau_\beta = B$, so that eq. (19) can be rewritten as $T_i = B/\tau_{3-i}$, and eq. (20) becomes,

$$\alpha_i = \frac{\tau_\alpha - \tau_{3-i}}{\tau_\alpha - \tau_\beta}, \qquad \text{and} \qquad \beta_i = -\frac{\tau_\beta - \tau_{3-i}}{\tau_\alpha - \tau_\beta}. \tag{31}$$

Finally, the voltage of electrode 2 follows from eq. (15) as,

$$V_2 = I_{R_2}R_2 = \frac{V_0R_2}{R_0 + R_1 + R_2}\left(1 - \frac{\tau_\alpha - \tau_1}{\tau_\alpha - \tau_\beta}e^{-t/\tau_\alpha} + \frac{\tau_\beta - \tau_1}{\tau_\alpha - \tau_\beta}e^{-t/\tau_\beta}\right). \tag{32}$$

A special case is that $R_1 = R_2 = R/2$ and $C_1 = C_2 = 2C$, which reduces to the example (sec. 2.1) of a single shell of resistance R and capacitance C. There are still two time constants from eqs. (25) and (30), $\tau_\alpha = RC$ and $\tau_\beta = R_0RC/(R_0 + R)$, of which τ_β is the one that appears in eq. (11). Then, according to eq. (20), $\alpha_i = 0$ and $\beta_i = 1$, so the time constant τ_α does not appear in the current.

Another special case is that $\tau_1 = R_1C_1 = R_2C_2 = \tau_2 \equiv \tau$. Then, eq. (20) tells us that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, and eq. (26) indicates that $\tau_\alpha = \tau_\beta = \tau R_0/(R_0 + R_1 + R_2)$. The current I has the form (12) where $R = R_1 + R_2$.

2.4 Oscilloscope and Probe

As shown in the lower figure on p. 1, an oscilloscope with a capacitive probe is an application of a pair of "leaky capacitors".

Here, we are particularly concerned with the response voltage, $V_2(\omega)$, at the oscilloscope input to a sinusoidal load voltage $V_0 e^{i\omega t}$. Noting that the impedance $Z_\parallel$ of a resistor R in parallel with a capacitor C is,

$$Z_\parallel = \frac{R}{1 + i\omega\tau}\,, \tag{33}$$

where $\tau = RC$, we have,

$$V_2(\omega) = IZ_2 = V_0 \frac{Z_2}{Z} = V_0 \frac{Z_2}{R_0 + Z_1 + Z_2} = V_0 \frac{R_2}{R_0(1 + i\omega\tau_2) + R_1 \frac{1+i\omega\tau_2}{1+i\omega\tau_1} + R_2}\,. \tag{34}$$

In practice, the resistances R_1 and R_2 should be large compared to R_0, in which case the probe acts as a frequency-independent voltage divider,

$$V_2 \approx V_0 \frac{R_2}{R_1 + R_2}\,, \tag{35}$$

provided $R_1 C_1 = \tau_1 = \tau_2 = R_2 C_2$.

2.5 Solution via Laplace Transforms

The frequency response (34) can be used to deduce the transient response to a constant voltage V_0 that is turned on at $t = 0$ via the techniques of Laplace transforms.

2.5.1 Definition of the Laplace Transform

Recall that in general a function $f(t)$ can be represented as a Fourier integral,

$$f(t) = \int_{-\infty}^{\infty} f_\omega e^{i\omega t}\, d\omega, \tag{36}$$

where the Fourier amplitude f_ω is given by,

$$f_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt. \tag{37}$$

For the special case that $f(t) = 0$ for $t < 0$, the Fourier amplitude is,

$$f_\omega = \frac{1}{2\pi} \int_{0}^{\infty} f(t) e^{-i\omega t}\, dt. \tag{38}$$

We can define $s = i\omega$, and introduce the Laplace transform $F(s)$ of the function $f(t)$ as,

$$F(s) = 2\pi f_\omega = \int_{0}^{\infty} f(t) e^{-st}\, dt, \tag{39}$$

in which case the Fourier integral (36) can be rewritten as,[1]

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{i\omega t}\, d\omega = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(s) e^{st}\, ds, \tag{40}$$

which relation is called the inverse Laplace transform.

[1]Strictly, the integration is along a contour in the complex s plane that includes a line along some value of $Re(s)$ such that all poles of $F(s)$ lie within the contour when it is closed to the left via an infinite half circle.

2.5.2 Analysis of Impulse Response

We now use the method of Laplace transforms for the example of oscilloscope and probe to determine the response waveform $V_2(t)$ from the driving waveform $V_0(t)$. A first approach follows the spirit of Green by supposing $V_0(t)$ is the Dirac delta function $\delta(t)$, *i.e.*, a brief impulse of voltage.

From eq. (38) we see that all Fourier components of $\delta(t)$ are the same, namely $1/2\pi$ since $\int \delta(t) f(t)\, dt = f(0)$. *From eq. (39) the Laplace transform of the delta function is 1.* Hence, dividing eq. (34) by $2\pi V_0$ gives the Fourier component of the response voltage $V_{2,\text{impulse}}$ caused by the impulse $\delta(t)$. Then, eq. (39) tells us that $V_2(\omega)$ is the Laplace transform of the response function, so we rewrite eq. (34) using $s = i\omega$ as,

$$\begin{aligned} F_{2,\text{impulse}}(s) &= V_2(\omega) = V_0 \frac{R_2(1+s\tau_1)}{R_0(1+s\tau_1)(1+s\tau_2)+R_1(1+s\tau_2)+R_2(1+s\tau_1)} \\ &= \frac{V_0 R_2}{R_0\tau_2} \frac{s+1/\tau_1}{(s-\sigma_\alpha)(s-\sigma_\beta)}\,, \end{aligned} \tag{41}$$

where σ_α and σ_β are the roots of the quadratic equation,

$$s^2 + \frac{R_0(\tau_1+\tau_2)+R_1\tau_1+R_2\tau_1}{R_0\tau_1\tau_2} s + \frac{R_0+R_1+R_2}{R_0\tau_1\tau_2} = s^2 + \frac{A}{B}s + \frac{1}{B} = 0, \tag{42}$$

where the constants A and B were introduced in eqs. (28)-(29). That is,

$$\sigma_\alpha = -\frac{A-\sqrt{A^2-4B}}{2B} = -\frac{1}{\tau_\alpha}\,, \qquad \text{and} \qquad \sigma_\beta = -\frac{A+\sqrt{A^2-4B}}{2B} = -\frac{1}{\tau_\beta}\,, \tag{43}$$

recalling eq. (30).

To find $V_{3,\text{impulse}}(t)$, we note that the inverse Laplace transform of the form,

$$F(s) = \frac{s+c}{(s+a)(s+b)} \tag{44}$$

is,

$$f(t) = \frac{(c-a)e^{-at} - (c-b)e^{-bt}}{b-a}\,. \tag{45}$$

Using this with eqs. (41) and (43), we obtain the form of the impulse response,

$$\begin{aligned} V_{2,\text{impulse}}(t) &= \frac{R_2}{R_0\tau_2} \frac{(1/\tau_1 - 1/\tau_\alpha)e^{-t/\tau_\alpha} - (1/\tau_1 - 1/\tau_\beta)e^{-t/\tau_\beta}}{1/\tau_\beta - 1/\tau_\alpha} \\ &= \frac{R_2}{R_0\tau_1\tau_2} \frac{\tau_\beta(\tau_\alpha-\tau_1)e^{-t/\tau_\alpha} - \tau_\alpha(\tau_\beta-\tau_1)e^{-t/\tau_\beta}}{\tau_\alpha-\tau_\beta}\,. \end{aligned} \tag{46}$$

The response $V_2(t)$ to any drive voltage $V_0(t)$ can be deduced from the impulse response,

$$V_2(t) = \int_{-\infty}^{t} V_0(t') V_{2,\text{impulse}}(t-t')\, dt'. \tag{47}$$

In particular, the response $V_{2,\text{step}}(t)$ to a step in voltage from 0 at $t, 0$ to V_0 for $t > 0$ is,

$$\begin{aligned}
V_{2,\text{step}}(t) &= V_0 \int_0^t V_{2,\text{impulse}}(t-t')\,dt' = V_0 \int_0^t V_{2,\text{impulse}}(t'')\,dt'' \\
&= \frac{V_0 R_2}{R_0 \tau_1 \tau_2}\left[\frac{\tau_\beta(\tau_\alpha - \tau_1)}{\tau_\alpha - \tau_\beta}\int_0^t e^{-t''/\tau_\alpha}\,dt'' - \frac{\tau_\alpha(\tau_\beta - \tau_1)}{\tau_\alpha - \tau_\beta}\int_0^t e^{-t''/\tau_\beta}\,dt''\right] \\
&= \frac{V_0 R_2 \tau_\alpha \tau_\beta}{R_0 \tau_1 \tau_2 (\tau_\alpha - \tau_\beta)}\left[(\tau_\alpha - \tau_1)(1 - e^{-t/\tau_\alpha}) - (\tau_\beta - \tau_1)(1 - e^{-t/\tau_\beta})\right] \\
&= \frac{V_0 R_2}{R_0 + R_1 + R_2}\left(1 - \frac{\tau_\alpha - \tau_1}{\tau_\alpha - \tau_\beta} e^{-t/\tau_\alpha} + \frac{\tau_\beta - \tau_1}{\tau_\alpha - \tau_\beta} e^{-t/\tau_\beta}\right), \qquad (48)
\end{aligned}$$

as found previously in eq. (32).

Equation (47) can be re-expressed in the language of Laplace transforms on noting that there is no response prior to the drive impulse, so $V_{2,\text{impulse}}(t) = 0$ for $t < 0$. Hence, we can extend the limit of integration in eq. (47) from t to ∞,

$$V_2(t) = \int_{-\infty}^{\infty} V_0(t') V_{2,\text{impulse}}(t - t')\,dt'. \qquad (49)$$

The Laplace transform $F_2(s)$ of eq. (49) is (assuming that $V_0(t) = 0$ for $t < 0$),

$$\begin{aligned}
F_2(s) &= \int_0^\infty e^{-st}\,dt \int_{-\infty}^{\infty} V_0(t') V_{2,\text{impulse}}(t-t')\,dt' \\
&= \int_{-\infty}^{\infty} V_0(t') e^{-st'}\,dt' \int_{-\infty}^{\infty} V_{2,\text{impulse}}(t-t') e^{-s(t-t')}\,dt \\
&= \int_0^\infty V_0(t') e^{-st'}\,dt' \int_0^\infty V_{2,\text{impulse}}(t'') e^{-st''}\,dt'' \\
&= F_0(s) \cdot F_{2,\text{impulse}}(s). \qquad (50)
\end{aligned}$$

Thus, the Laplace transform $F_{2,\text{impulse}}(s)$ of the response $V_{2,\text{impulse}}(t)$ to a drive impulse at $t = 0$ equals the Laplace transform $F_2(s)$ of the response $V_2(t)$ to any drive waveform $V_0(t)$ (provided $V_0(t) = 0$ for $t < 0$) divided by the Laplace transform $F_0(s)$ of that drive waveform,

$$\frac{F_2(s)}{F_0(s)} = F_{2,\text{impulse}}(s) \equiv \textsf{transfer function}. \qquad (51)$$

The ratio of the Laplace transform of the response function to the Laplace transform of the drive function is called the transfer function*. Thus, eq. (41) describes the transfer function for the present example.*

2.5.3 Step Response via the Impulse Response Transform

The response to a step driving term can also be found by a slightly different procedure. We note that the delta function $\delta(t)$ is the time derivative of the Heaviside step function,

$$\delta(t) = \frac{d\theta}{dt}, \qquad \text{where} \qquad \theta(t) = \begin{cases} 0 & (t < 0), \\ 1 & (t > 0). \end{cases} \qquad (52)$$

We also see from eq. (39) that the Laplace transform of the time derivative df/dt is s times the Laplace transform of function $f(t)$,

$$\int_0^\infty \frac{df}{dt} e^{-st}\, dt = f e^{-st}\Big|_0^\infty + s \int_0^\infty f e^{-st}\, dt = sF(s). \tag{53}$$

Hence, the Laplace transform of the step response is equal to the Laplace transform of the impulse response divided by s. *Recalling the discussion at the end of sec. 2.5.2, this relation can also be stated as: the Laplace transform of the step response is equal to the transfer function divided by s.*

On dividing the transfer function (41) by s, the Laplace transform of the response $V_{2,\mathrm{step}}$ to a step in voltage V_0 is

$$\frac{V_0 R_2}{R_0 \tau_2} \frac{s + 1/\tau_1}{s(s + 1/\tau_\alpha)(s + 1/\tau_\beta)}\,. \tag{54}$$

The inverse of the Laplace transform,

$$\frac{s + c}{s(s + a)(s + b)} \tag{55}$$

is,[2]

$$f(t) = \frac{c}{ab} \left(1 - \frac{b(c - a)}{a - b} e^{-at} + \frac{a(c - b)}{a - b} e^{-bt} \right). \tag{56}$$

The particular form (54) then leads immediately to the step response (32) and (48).

[2]See, for example, D. Christiansen, R. Jurgen and D. Fink, *Electronics Engineers' Handbook*, 4$^{\rm th}$ ed. (McGraw-Hill, 1996).