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From the Editor's Desk...

This month's word is CAVEAT. From the Latin: let him beware; a warning. In the January issue we published a letter from JACG member William Hough concerning his unfortunate luck with two Astra disk drives. We have since received a packet of materials from Astra Systems, Inc. letting us know of their distress at our printing of "Mr. Hough's mendacious assertions." The letter, from Astra plant manager Drew Featherston, takes us to task for publishing such a "libelous and defamatory letter" and chastises us for not verifying the facts and allowing Astra to present their case.

In the second case we are most guilty. In the fairness of journalism it was, indeed, a professional discourtesy not to invite Astra to counter Mr. Hough's charges. For that we apologize. Being a small publication intended for distribution to our membership and some other user groups we never have been so egocentric to think of ourselves in the same league as ANTIC or ANALOG magazines. Mr. Featherston, however, points out that these publications refused to print Mr. Hough's "malicious missive" because they "recognized their legal obligations."

A review of the materials from Astra makes it apparent that there was considerable mix-up in communications between Astra and Mr. Hough. Astra apparently tried to set things straight but the corporate-customer relationship had too far soured. In all fairness, it further seems that Astra has cured the basic problem of excessive overheating. Copies of letters from satisfied customers indicate an excellent relationship with Astra.

In the final analysis it seems what we have here is the sort of unfortunate situation described by Tom Peters in his popular book, "In Search of Excellence", wherein the true human sensitivity of a corporation can get lost in administrative procedures. This is not to say either side in this dispute was/is right or wrong. The perceptions are what count.

As the official organ of our group we feel it is our responsibility to report on Atari-relative matters. That is why we published the letter originally. We had received many reports of Astra drives failing. We are sure that Mr. Featherston (correctly) feels it his right and responsibility to protect the good name of Astra Systems, Inc. Again, in fairness, it seems that Astra is doing everything they can to rectify these past problems and are now producing an effective and dependable product. We would invite you to look through the dossier of materials Mr. Featherston sent and inform yourself. See any club officer for a copy of these materials.

In the future we certainly will make it a point to continue to print both positive and negative criticism of products which our membership might come in contact with. We will also make it a point to invite differing and opposing views. At the risk of appearing sloppily sentimental, we think that is the American way.

Briefly, another caveat should be observed as you peruse this colorful issue. It is April, after all, and those who have been among us more than a year know the danger that lurks within.

Frank Pazel

Frank Pazel
Editor-in-Chief, JACG Newsletter

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LEAF STORM

by Kirk McDonald - JACG

An exciting challenge for the computer graphics enthusiast is the creation of images which are "life-like." The difficulty is that a computer is most easily programmed to produce pictures which are either highly ordered, or completely random. But life is found in a fertile realm somewhere between the crystals and the clouds, and not necessarily in computer code.

One approach to life-like imagery is the use of a graphics input device such as the Koala Pad. In this case the computer merely digitzes a picture created by a human being, and plays almost no direct role in the creative process.

A more ambitious goal is a computer program which draws life-like pictures on the basis of a very small number of initial parameters specified by the user. An important step in this direction has been made by Mehrdad Shahshahani of Boeing Aerospace Co.

[A brief report on his work appeared on page 494 of the August 3, 1984 issue of Science magazine. A highly mathematical discussion of the basis of the algorithm has been given by Persi Diaconis and Shahshahani in Technical Report No. 228 of the Stanford University Department of Statistics, dated November 1984.]

You can explore the strange and wonderful insight of Shahshahani on your Atari 800 or 800XL computer with the aid of the program LEAF STORM. (The name is borrowed from a short story by Garcia Marquez.) This is available on a JACG library diskette in two forms. The source code is in file LEAF.ACT and is written in the Action! language. The binary file LEAF.COM can be run directly from DOS if you do not have an Action! cartridge. The file LEAF.DOC is a text file which contains some instructions on running the program if you do not find it self-explanatory. This file also contains a short description of the mathematics of the program.

To generate a picture you must provide the computer with a list of "transformations." Each transformation is just a set of four numbers. If you specify only one transformation the computer will plot exaclty one dot. But already if you enter two transformations (8 numbers) the computer can draw quite interesting shapes. Figure 1 was produced by entering

```
50 -40 -30 0
50  20 -30 0
```

To my forgiving eye this looks something like the outline of a clump of trees, or a cloud-bank. Of course, we must heed Shakespeare's remark that a cloud may appear simultaneously as a camel, a weasel and a whale to the impressionable.

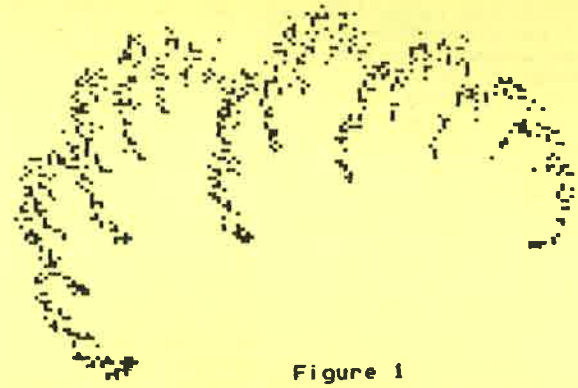


Figure 1

Figure 2 was drawn with the 3 transformations

```
58 30   0 -30
58 70  30   0
58 10 -30   0
```

Figure 2

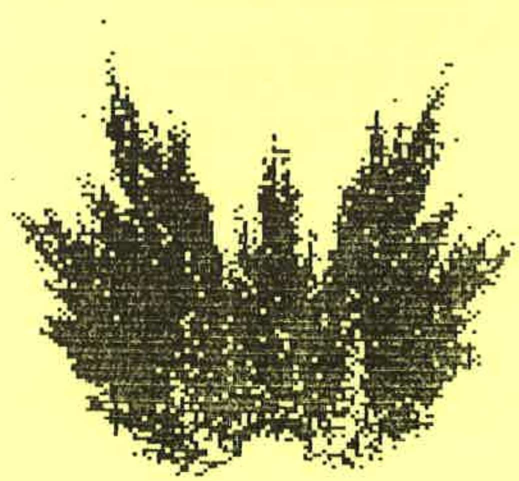


Figure 3

Figure 3 is an example of the eponymous leaf. Its transformations are

50	-43	-31	0
50	46	-29	0
50	-20	-5	0
50	22	-5	0

Figure 4 was generated by Shahshahani and is called a poplar treee by him.

Figure 4

These figures do not show a curious aspect of their generation, which would be apparent if you watch them being drawn on your TV screen. The computer very quickly draws a rough outline of the image, and then returns to fill in with greater detail. If the TV screen had infinite resolution, and the computer was left the draw forever, it would add ever finer shading to the picture, without altering the overall form to any great extent. This behavior is closely related to fractals, which are known to be associated with interesting graphics. To my taste, most fractal pictures are rather crystalline, while Shahshahani has succeeded in combining crystals and clouds in subtle proportion so as to simulate life.

Figure 5 illustrates the fractal aspects of the LEAF STORM. The shrimp-like image is generated by the 2 transformations

55	10	34	34
55	-34	-21	-33

Figure 5

Don Forbes
8 Birch Lane
Florham Park N.J
07932

The LEAF STORM algorithm may give us a glimpse as to how complex forms of life can be compactly encoded in DNA molecules. As the pictures are drawn by the computer one gets the feeling of witnessing an evolution from a primitive form to a more and more refined structure. Furthermore, a small "mutation" of the transformation parameters leads to slightly altered images. However, the alteration is not localized to a small region of the image, but is distributed over the whole in a complex way. This simulates the manner in which various members of the same species have a common overall form but differ in detail.

The mutation of the transformations can be readily applied to generate computer animation. Figure 6 shows a somewhat stylized man (Kachina doll?) taking a step. The starting set of transformations is

30	10	60	0
30	-10	60	0
50	30	-10	0
50	-30	-10	0

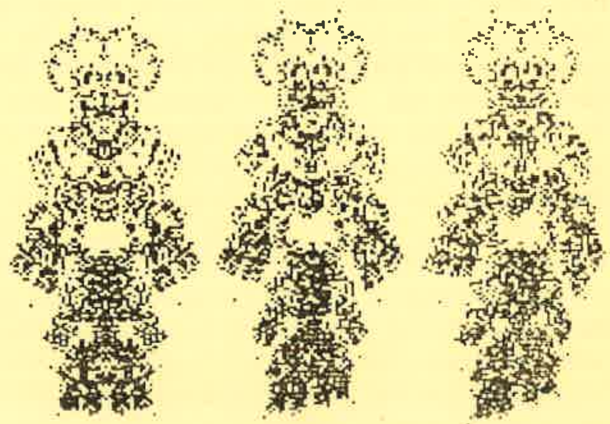


Figure 6

The 2nd step was obtained by changing the two values of 30 to 31 and 29, while for the 3rd step the values were 32 and 28.

Perhaps Walt Disney Studios is in no immediate danger of being replaced by an Atari 800 computer. But when the 520ST becomes available.....

overlying the gently inclined subducting ocean crust. A section of ocean crust apparently broke off and underlies the accreted terranes. Prominent reflectors in recently acquired data coincide with the upper-lower crust boundary, the bottom of the detached oceanic plate, and either the continental Moho or the top of the subducting ocean plate.

While the USGS and non-U.S. programs emphasize the use of multiple techniques at the same sites, COCORP speakers placed a new emphasis on the reconnaissance nature of most of their program. Refraction will help, but the ultimate test of seismic methods—deep drilling—is only now being proposed. A 10-kilometer hole in the southern Appalachians (*Science*, 29 June, p. 1418) would help determine what some reflectors really are.—RICHARD A. KERR

Esoteric Math Has Practical Result

A new method of computer graphics relies on math results that seemed so abstruse that they were never published

The problem with natural objects is that they are so irregular. When programmers try to tell a computer how to draw a cloud or a leaf or a forest, they run into difficulties. If they attempt to specify each and every detail, they will come up against a monumental computing task. It can take thousands or millions of bits of stored data to draw a realistic scene and computers quickly run out of space. It also takes computers a long time—18 hours in some cases—to put all this stored data together to make a picture. If programmers try to provide general rules for drawing scenery, the computer pictures will look a little too smooth and regular. It is even more difficult to solve what computer scientists call the encoding problem. Take a scene, digitize it, and compress the information substantially so it can be easily stored. Then ask the computer to recreate that exact scene any time you want to see it.

But Mehrdad Shahshahani, a mathematician at Boeing Aerospace in Seattle has an extremely promising approach to solving both of these problems. He has found a way to make computer pictures of natural objects and to encode pictures of scenery with very little effort. He can generate a realistic picture of a leaf, for example, with only 21 numbers and three simple equations. Boeing Aerospace wants to use Shahshahani's results in its flight simulators, which are computerized systems used to train pilots by giving them the exact sensations of flying, complete to the scenery outside the window, without ever leaving the ground.

Shahshahani's work relies on some very esoteric abstract mathematics, which seems so unrelated to the real world that when Persi Diaconis, a statistician at Stanford University, studied this math 10 years ago, he decided not even to publish his results. But the mathematics results turned out to be just what is needed to determine which numbers and simple equations will make which pictures of natural objects. The story of how this mathematics came to light is the sort of story that is dearest to mathematicians' hearts. It is a story of mathematics pursued for its own ends that eventually finds an unexpected and significant use.

In 1974, as a graduate student in the statistics department of Harvard University, Diaconis at first had difficulty finding a research problem that interested him. Then, by chance, he came upon the "first digit problem," a problem first described around the turn of the century by Simon Newcomb, an astronomer. Newcomb was led to a curious result about the distribution of the first digits of numbers when he noticed that the beginning pages of books of logarithms were the most worn, indicating that people were looking up more logarithms of numbers starting with 1 than any other number.

If you look at the lead digit in any source of numbers, such as the pages of *Science* or the numbers in the almanac, you might expect that the number 1 would turn up about one-ninth of the time. After all, there are nine possible first digits and there is no reason to believe that any one digit would be favored over any other. But, surprisingly, the number 1 is the first digit about three times out of ten because the numbers that begin with 1 are irregularly spread among all the numbers. So, for example, one-ninth of the numbers from 1 to 9 begin with 1. One-half of the numbers from 1 to 20 begin with 1. One-ninth of the numbers from 1 to 100 start with 1. One-half of the numbers from 1 to 200 start with 1. As you look at larger and larger sets of numbers, the proportion of numbers in the sets with lead digits that are 1 oscillates between one-half and one-ninth. Diaconis asked whether there was some other natural way to take an average so that the average number of lead digits that are 1 will settle down rather than oscillate.

A way to do this, Diaconis found, is to use the Riemann zeta function, which has been the object of intense study for the past century because if more were known about it, more would be known about where prime numbers lie. The zeta function is an infinite sum, and Diaconis found that if he used the terms of that sum as weighting factors, he could get a way of averaging that would avoid the oscillations in the first digit problem. At the same time, this averaging method would give the usual sort of average in cases where the average does not oscillate. For example, both it and the ordinary way of taking an average say that one-half of all whole numbers are even. His method of "zeta averages" says that the density of the set of numbers that begin with 1—the chance that if you pick a number at random it will begin with 1—is $\log_{10}2$, or .301.

"This was very esoteric math," Diaconis says. "It was the sort of math that made people say, 'Gee, that's funny, but why would anyone care?' " In fact, when Diaconis went to the University of California at Berkeley in 1973 to give a talk on his thesis as part of a job interview, he recalls the Berkeley statisticians saying, "We assume you'll find something else to work on."

Diaconis accepted a job at Stanford rather than at Berkeley and he did find many other things to work on. He all but forgot the first digit problem. In the meantime, Shahshahani was investigating a highly innovative way of producing computer graphics. He got his inspiration from some work done 3 years ago by John Hutchinson of the Australian National University in Canberra. Hutchinson was interested in generating fractals, which are mathematical entities with fractional dimensions. "There was no indication that he wanted to use his method for pictures," Shahshahani says. But Shahshahani recognized that Hutchinson's ideas could be used for computer graphics.

What Shahshahani does is to take very ordinary curves and lines and repeatedly apply certain simple transformations, called affine transformations, which deform them. Then he looks at the fixed points, which are those that do not move under the transformations. By plotting the fixed points he generates pictures of natural objects.

Shahshahani came to Diaconis to gain more insight into why the procedure works and how to best choose the original transformations. Diaconis looked at the problem. "I sat there and shook my head. The way to solve the problem is to use exactly the crazy results from my thesis that I never published."

Shahshahani's method is to start with a few particular affine transformations, which are operations that take a line in the plane and first contract and rotate it and then shift it to a new position. He applies the first transformation to a line and plots the fixed point. Then he applies the second transformation and plots the fixed point. Next he multiplies the first transformation times the second, applies the resulting transformation to the line, and plots the fixed point. He then multiplies the second transformation times the first (multiplication of transformations is not commutative so the product of the first and the second is different from the product of the second and the first) and repeats the process of plotting the fixed point.

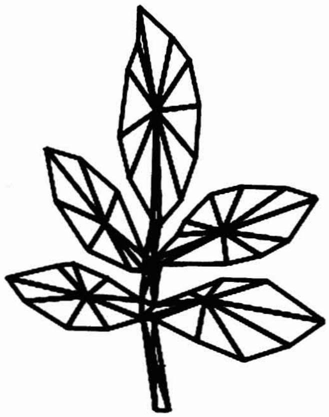
The next step is to multiply every combination of the original transformations when they are grouped in threes, such as $1 \times 2 \times 1$, $2 \times 1 \times 1$, $1 \times 1 \times 2$, and so on. Once again, these resulting product transformations are applied to the line and the fixed points are plotted. Shahshahani stops the process when he gets enough points for a realistic picture.

When Diaconis looked at Shahshahani's procedure, he recognized a math problem that can be expressed in the language of Markov chains—sequences of events with the property that each event in the chain depends only on the preceding one. They are widely used in statistics to model random phenomena.

In Shahshahani's case, the Markov chain was the sequence of affine transformations. The process of picking the transformations in the sequence gives the same result whether it is done deterministically or randomly. For example, you could flip a coin to decide whether to start with transformation 1 or 2. Then when you are multiplying groups of two transformations together, you could flip a coin to decide what order to multiply them in.

The fundamental theorem in Markov chain theory says that eventually the chain has to settle down into a stationary distribution. For example, Diaconis says, if he is mixing the cards in a deck and he starts with the cards in a known order, he may proceed by switching two cards at random. Then he may switch two other cards. As he continues in this way, the arrangement of cards in the deck will get more and more random. In that case, the stationary distribution is a random mixture of cards.

To find what initial transformations will lead to various stationary distributions required analyzing random infinite sums—the sums being the transformations. "It was precisely like the sums in my thesis," Diaconis remarks. "I called Mehrdad and said, 'I know all about these things!' " Based on Diaconis' results from analyzing the first digit problem, he and Shahshahani can predict stationary distributions and so are able to tell what computer pictures will look like on the basis of the original transformations. More importantly, they can solve the harder "inversion" problem of telling what transformations will give rise to particular pictures of natural objects.



The new and the old

The leaf at left was drawn with the new "fixed point" method. It required 12 numbers for the stem and 21 for the main part of the leaf. The leaf at right was drawn in the conventional way, by specifying the lines for the computer to draw. [Sources: Merhdad Shahshahani (left); Cranston/Csuri Productions, Columbus, Ohio (right)]

Once Diaconis expressed Shahshahani's problem in the language of Markov chains, he saw that what Shahshahani really wanted to know was the stationary distributions he will get with various initial transformations. That would tell him how his choice of affine transformations affects the computer picture.

"The quality that Mehrdad's pictures have is that they are 'leafy.' They don't smear out over the screen. They have a delicate fine structure no matter how closely you look," Diaconis remarks. "The main issue was to determine the mathematical notion that captures this property." Diaconis believes that the key to leafyness is a notion of being "singular continuous," a mathematical phenomenon, Diaconis says, "that was always thought to have no application whatsoever." Shahshahani disagrees, arguing that it is other aspects of the distribution that are important. Both ideas, however, can be investigated with the research the two are undertaking.

Shahshahani is developing computer programs to implement the inversion procedure. Now, he says, that "the math is under control," he expects within a few months to be able to make pictures of simple natural objects, digitize them, and reproduce them exactly on the basis of very little stored information. This will be "not just a leaf or a tree that looks nice; you tell me what kind of leaf you want and I should be able to generate it."

Diaconis, in the meantime, is rereading some papers on random infinite sums published in the 1930's by the Hungarian mathematician Paul Erdös. "Erdös studied them for no reason other than that they were beautiful and interesting. At the moment, I'm going over Erdös's papers with a fine tooth comb," Diaconis remarks. It looks as if the more that is known about these seemingly esoteric quantities, the more will be known about how to encode and generate computer pictures of natural objects.

—GINA KOLATA

Department of Statistics

PRODUCTS OF RANDOM MATRICES AND COMPUTER IMAGE GENERATION

BY

PERSI DIACONIS AND MEHRDAD SHAHSHAHANI

TECHNICAL REPORT NO. 228
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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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PRODUCTS OF RANDOM MATRICES AND

COMPUTER IMAGE GENERATION

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§1 Introduction.

Fractals as models for certain natural phenomena were defined and popularized by Mandelbrot ([M]). Subsequently, Mandelbrot and others produced realistic pictures of mountains, landscapes, clouds etc. using fractal methods. In [H], Hutchinson proposed a beautiful mathematical framework for the study of fractals (see §2). We noticed that this method can be used for computer generation of pictures of certain natural objects while acheiving substantial data compression. For example, the poplar tree below was generated by storing only 52 numbers. The question naturally arose as to whether the fixed-point method of Hutchinson can be used in a systematic fashion for computer image generation and data encoding. It became clear that further theoretical development is necessary to apply this procedure in a systematic manner. In this paper we give a brief account of some of our results in this direction.

We conclude this discussion with a brief description of the algorithm that produced Figure 1. The algorithm uses as input n affine transformations $\{S_1, \ldots, S_n\}$. Here $S_i x = A_i x + c_i$ with x and $c_i \in \mathbb{R}^2$ and A_i a 2×2 matrix with $|A_i x| < |x|$ for all $x \neq 0$. Such a transformation has a unique fixed point; from $S_i x = x$ we have $x = (I - A_i)^{-1} c_i$. The first stage of the algorithm plots the n fixed points. The second stage plots the fixed points of all n^2 products $S_i S_j$. At stage k, all n^k fixed points of products of length k are plotted. The algorithm terminates when sufficiently many points have been plotted.

For the picture in Figure 1, the leaf is based on $n = 4$, $k = 7$ and for the stem we have $n = 3$, $k = 7$.

§2 Fractals.

We first review the relevant aspects of the mathematical framework proposed by Hutchinson [H] for the study of fractals. Let $\Delta = \{S_1,\ldots,S_n\}$ be a finite set of contracting affine transformations of $\mathbb{R}^d$. Then every transformation $S_{i_1} \ldots S_{i_k}$ has a unique fixed point which we denote by $F_{i_1\ldots i_k}$. The fractal object associated to Δ is

$$F[\Delta] = \text{closure } \{F_{i_1\ldots i_k} \mid \text{all } k \text{ and all } i_1,\ldots,i_k\} .$$

This is the "limiting picture" produced by the algorithm of Section 1.

THEOREM 2.1 – ([H]).

(a) $F[\Delta]$ is the unique compact subset K of $\mathbb{R}^d$ with the property

$$K = S_1(K) \cup\ldots\cup S_n(K) .$$

Let

$$\Omega = \Pi\{1,\ldots,n\}$$

be the set of all mappings of $\mathbb{N}$ into the finite set $\{1,\ldots,n\}$ equipped with the product topology.

(b) For an element $w = (i_1,i_2,\ldots) \in \Omega$ and any $x \in \mathbb{R}^d$,

$$\lim_{k\to\infty} S_{i_1} S_{i_2} \ldots S_{i_k}(x)$$

exists and is independent of x. We denote by $\pi(w)$ this limiting value, and so we have a map

$$\pi:\Omega \to \mathbb{R}^d .$$

(c) The map π is continuous and

$$\operatorname{Im} \pi = F[\Delta] .$$

It is not difficult to see

<u>Corollary</u> - $F[\Delta]$ is a perfect set unless all S_j's have the same fixed point F, in which case $F[\Delta] = \{F\}$.

It is clear that understanding the structure of the set $F[\Delta]$ is a fundamental problem in this area. There is an important special case where we can say more about $F[\Delta]$.

<u>Condition SC</u> - We say condition SC (strongly contracting) holds if there is an open set $U \subset \mathbb{R}^d$ and disjoint compact neighborhoods $V_1, \ldots, V_n$ of $F_1, \ldots, F_n$ such that

$$S_j(U) \subset V_j, \quad \bigcup_{j=1}^{n} V_j \subset U$$

THEOREM 2.2 - Assume (SC) holds.

(a) π is a homeomorphism onto $F[\Delta]$.

(b) Suppose furthermore S_j's are similitudes (so $S_j = r_jA_j+c_j$ with A_j orthogonal) with contraction rates r_j respectively. Then

$$\text{Hausdorf dimension } F[\Delta] = D$$

where D is the unique real number such that

$$\Sigma r_j^D = 1 ,$$

((b) is proved in [H].)

Let (X,d) be a compact metric space and $\varepsilon > 0$. Define

$N(\varepsilon)$ = smallest number of balls of radius ε covering X

$M(\varepsilon)$ = largest integer m such that there are m points $x_1,\ldots,x_m \in X$ with $d(x_i,x_j) \geq \varepsilon$ for $i \neq j$.

The limits

$$\lim_{\varepsilon\to 0} \frac{\log M(\varepsilon)}{-\log \varepsilon} = C \,, \qquad \lim_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = H \,,$$

if they exist, are called <u>capacity</u> and <u>metric entropy</u> of X. Let ρ_i be the operator norm of S_i, and η_i be the largest number such that

$$|A_i x| \geq \eta_i |x|$$

for all $x \in \mathbb{R}^d$. Set

$$\rho = \max(\rho_1,\ldots,\rho_n) \,,$$

$$\eta = \min(\eta_1,\ldots,\eta_n) \,.$$

For the metric space $F[\Delta]$ with induced metric from $\mathbb{R}^d$ we show

<u>Proposition 2.3</u>.

(a)
$$\overline{\lim} \frac{\log N(\varepsilon)}{-\log \varepsilon} \leq \frac{\log n}{-\log \rho} \,.$$

(b) If condition (SC) holds, then

$$\frac{\log n}{-\log \eta} \leq \underline{\lim} \frac{\log M(2\varepsilon)}{-\log \varepsilon} \,.$$

Furthermore, if S_j's are contractions with the same contraction rate ρ, then the capacity and metric entropy of $F[\Delta]$ exist and are equal to the Hausdorf dimension

$$\frac{\log n}{-\log \rho}\ .$$

§3 The Associated Markov Chain.

Let $p_1, \ldots, p_n$ be positive numbers with $\Sigma\, p_j = 1$, and P be the probability measure on the affine group $A(d) = \mathbb{R}^d$, $GL(d,\mathbb{R})$ (semi-direct product) supported on $\Delta = \{S_1,\ldots,S_n\}$ and assigning mass p_j to S_j. Consider the Markov chain $\{X_n\}_{n=0,1,\ldots}$ on $\mathbb{R}^d$ determined by the starting state $X_0 = x_0$ and

$$X_{n+1} = A_{n+1}\, X_n + c_{n+1} \qquad\qquad n \geq 0$$

with (c_n,A_n) i.i.d. from P. The basic properties of this Markov are summarized in

THEOREM 3.1. (See [DB] and [H]).

(a) The Markov chain $\{X_n\}_{n=0,1,\ldots}$ has a unique stationary distribution represented by the sum

$$S = c_1 + A_1c_2 + A_1A_2c_3 + \ldots$$

with (c_n,A_n) i.i.d. from P.

(b) The law of μ is of pure type. It is continuous unless all S_j's have the same fixed point F, in which case S is atomic concentrated at F.

(c) The law of S satisfies

$$E\{f(S)\} = \int E\{f(AS+c)\}\, P(dc,dA)$$

for every Borel function f.

(d) $\mathrm{Supp}(\mu) = F[\Delta]$.

The stationary distribution μ is a mathematical representation of the picture we see on the monitor. To make this relationship precise, let E_k denote the empirical measure of the set of fixed points of all words of length k (counted with multiplicities). Then it is not difficult to show

Proposition 3.2.

(a) E_k converges to μ in the weak star topology.

(b) $\cup_{j=1}^{k}$ Supp E_j converges to $\mathrm{supp}(\mu)$ in the Hausdorf metric, and $\mathrm{supp}(\mu)$ varies continuously with (c_i, A_i).

We equip Ω with the usual cylinder measure λ which is the unique exchangeable probability measure on Ω where

$$\lambda(\{(j_1,\ldots,j_k,* * \ldots)\}) = p_{j_1} \cdots p_{j_k} .$$

The following theorem is a version of the law of large numbers; it implies that the points hit by the Markov chain can be approximated by μ.

THEOREM 3.3 - Assume condition (SC) holds, and $B \subset \mathbb{R}^d$ is an open set satisfying

$$\bar{B} \cap F[\Delta] = B \cap F[\Delta]$$

Let $x \in \mathbb{R}^d$ and for $\omega = (i_1, i_2, \ldots) \in \Omega$ let

$$N(\omega,k) = \mathrm{Cardinality}\{(i_1,\ldots,i_j) \mid j \leq k,\ S_{i_j} \ldots S_{i_1}(x) \in B\} \quad .$$

Then there is Ω' of λ-measures 1, and independent of B, such that for all $\omega \in \Omega'$

$$\mu(B) = \lim_{k\to\infty} \frac{N(\omega,k)}{k} .$$

This theorem enables us to give a dynamical systems interpretation to the Markov chain. Assume condition (SC) holds and choose $\omega = (i_1, i_2, \ldots)$ for which the conclusion of the theorem holds. Consider the time dependent difference scheme

$$T_j(x) = S_{i_j}(x) .$$

Then $F[\Delta]$ is the (strange) attractor for this dynamical system. This also gives a partial answer to a question rasied by Guckenheimer about what kind of fractals can appear as strange attractors, [Gu].

It is instructive to compare Theorem 3.3 with conclusions easily derived from the ergodic theorem. The ergodic theorem implies that for almost all x in the support of μ and almost all sample paths, the empirical measure converges weak* to μ. Theorem 3.3 concludes this for any starting state under the hypothesis (SC). Consideration of simple examples such as $x \to \frac{1}{3}x$, $x \to \frac{1}{3}x + \frac{2}{3}$ shows that points outside $F[\Delta]$ may never get into $F[\Delta]$.

Understanding the stationary measure is a fundamental problem. Clearly if (SC) holds then μ is singular. There are a number of isolated results regarding absolute continuity of μ, for example, Erdös [E] has shown

(a) If $d = 1$, $n = 2$, $p_1 = p_2 = \frac{1}{2}$,

$$S_1(x) = ax+1 , \qquad S_2(x) = ax-1$$

and $a = 1/\zeta$ where ζ is a P.V. number, then μ is singular continuous. (There are infinity of P.V. numbers in the interval $(1,2)$.)

(b) For almost all a sufficiently close to 1, μ is absolutely continuous. See also [G] for conditions ensuring absolute continuity of μ.

The techniques of Erdös and Garcia can be generalized to higher dimensions to yield some conditions for singularity or absolute continuity for μ. This gives a nice application: mathematics invented for a very different purpose being used for a problem in computer graphics.

§4 Inverse Fractal Problem.

For the practical problem of image generation and data encoding the inverse fractal problem is the most fundamental question. In its simplest form it may be formulated as follows:

IFP1 - Given a compact set $X \subset \mathbb{R}^d$, does there exist a finite set $\Delta = \{S_1,\ldots,S_n\}$ of contracting affine transformations such that $F[\Delta]$ approximates X to given accuracy, say in the Hausdorf metric?

If

$$\overline{\lim_{\varepsilon\to 0}} \frac{N(\varepsilon)}{-\log \varepsilon} = H$$

exists, then it is easy to see that for every $\delta > 0$, a Δ exists such that

$$d(X,F[\Delta]) < \delta .$$

However, in this solution the cardinality of Δ is of the order of δ^{-H} as $\delta \to 0$, so that such a solution has small practical implication. For the special case of a convex polytope one has the following simple solution:

Proposition 4.1. Let X be a convex polytope with n vertices. Then there is a set $\Delta = \{S_1,\ldots,S_n\}$ of n contracting affine transformations such that

$$X = F[\Delta] .$$

In this proposition we take S_j's to be of the form

$$S_j(x) = \alpha_j Ix+c_j \qquad \alpha_j \in \mathbb{R}, \quad c_j \in \mathbb{R}^d$$

with α_j's sufficiently close to 1 and $\{F_j\}$ is the set of vertices of the convex polytope X. Then, the polytope becomes the attractor K of Theorem 2.1 which yields the desired result.

If we replace α_j's by general linear transformations, then we do not know of any simple extension of Proposition 4.1. However, it is important to note that the fractal object $F[\Delta]$ changes continuously as we vary the parameters. This fact is quite significant in the practical applications of the fixed point method. Furthermore, we can use the Markov chain introduced in §3 to propose a solution to the following more precise reformulation of IFPI:

IFP2 - Given a compactly supported probability measure α on $\mathbb{R}^d$, does there exist a finite set $\Delta = \{S_1,\ldots,S_n\}$ of contracting affine transformations such that the stationary measure associated to Δ (for some fixed $p_1, \ldots, p_n$) approximates α to given accuracy say in a Vasershtein metric?

For simplicity of notation, we describe our proposed solution to this problem only for $d = 1$. Note that Theorem 3.1 (c) is equivalent to

$$* \qquad \int f(x)d\mu(x) = \Sigma\, p_j \int f(S_j x)d\mu(x)$$

where μ is the stationary measure. Substituting

$$f(x) = x^k$$

in (*) we see that

$$\text{l.h.s.}(*) = m_k(\mu)$$

i.e., k^{th} central moment of μ. Evaluating r.h.s. (*), setting it equal to l.h.s. and simplifying we get

$$(**) \qquad m_k(\mu) = \frac{\sum_{j=1}^{n} p_j \sum_{r=1}^{k} \binom{k}{r} a_j^{k-r}\, c_j^r\, m_{k-r}(\mu)}{1 - \sum_{j=1}^{n} p_j\, a_j^k}\,.$$

Now given a compactly supported probability measure α on $\mathbb{R}$, we calculate its first $2k(k>1)$ moments. Then in principle we can solve the system of equations (**) to obtain a_j's and c_j's , so k affine transformations whose stationary measure has the same first $2k$ moments as α. It is not difficult to show that relative to the Vasershtein metric (see e.g. [BF] for definition and basic properties), the distance between α and μ tends to 0 as $\frac{c}{\sqrt{k}}$.

The calculation above can be extended to higher dimensions, but the algebraic manipulations are significantly more complex. A similar method for iteration of maps had been developed earlier by M. Barnsley and Demko [B]. The applicability of this procedure to actual problems has not yet been fully tested.

§5 Some Open Problems.

Here we briefly indicate a few open problems.

(a) As noted earlier, under rather restrictive hypothesis one can establish the existence of capacity and metric entropy for the fractal $F[\Delta]$. One may conjecture that these quantities always exist for such sets. A number of numerical invariants, such as information dimension, are defined in [FOY]. Computing these invariants for $F[\Delta]$ is an interesting problem.

(b) In view of Theorem 2.2(a), if condition (SC) holds, then $F[\Delta]$ is totally disconnected. Little is known about the topological structure of $F[\Delta]$. Consider for example the case of 3 contracting affine transformations S_1, S_2, S_3 of $\mathbb{R}^2$. Write

$$S_j(x) = A_j x + c_j$$

and assume S_j's have distinct fixed points $\{F_j\}$ not lying on the same line. If

$$A_j x = \alpha_j x \qquad \alpha_j \in \mathbb{R} ,$$

and

$$\Sigma\alpha_j \geq 2$$

then $F[\Delta]$ is the triangle E with vertices F_1, F_2, F_3. If $0 < \alpha_j < 1/2$, then condition (SC) is satisfied and so $F[\Delta]$ is totally disconnected. We can show that in intermediate cases where α_j's are such that $\alpha_j+\alpha_k > 1$ for all j,k, and $\alpha_1+\alpha_2+\alpha_3 < 2$, $F[\Delta]$ is path-wise connected, and its first homotopy group $\pi_1(F[\Delta])$ is infinitely generated. Clearly one may conjecture many generalizations of this result.

(c) The stationary measure μ is a fundamental quantity both from the theoretical and practical standpoints. Unfortunately, we only know of some isolated results about the nature of μ. One possible approach for establishing absolute continuity of μ in some cases is as follows: Define the operator T on $L^1(\mathbb{R}^d)$ by

$$T(\phi)(x) = \Sigma\, p_j(\det A_j)\ \phi(S_j x)$$

where $S_j x = A_j x + c_j$. Given any probability density ϕ, the average

$$\frac{1}{m}\sum_{j=0}^{m-1} T^j(\phi) = F_m(\phi)$$

converges weakly to the stationary distribution μ. If one shows that the sequence $\{F_m(\phi)\}$ is of bounded variation, then one can deduce absolute continuity of μ. This idea has been successfully used in the theory of iteration of maps (see e.g. [LY]), and so one may expect it to be applicable to this case too.

Arguments based on Fourier analysis (such as in [E1] and [E2]) are another possible approach to understanding the nature of the measure μ.

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