Can the Field Lines of a Permanent Magnet Be Tied in Knots?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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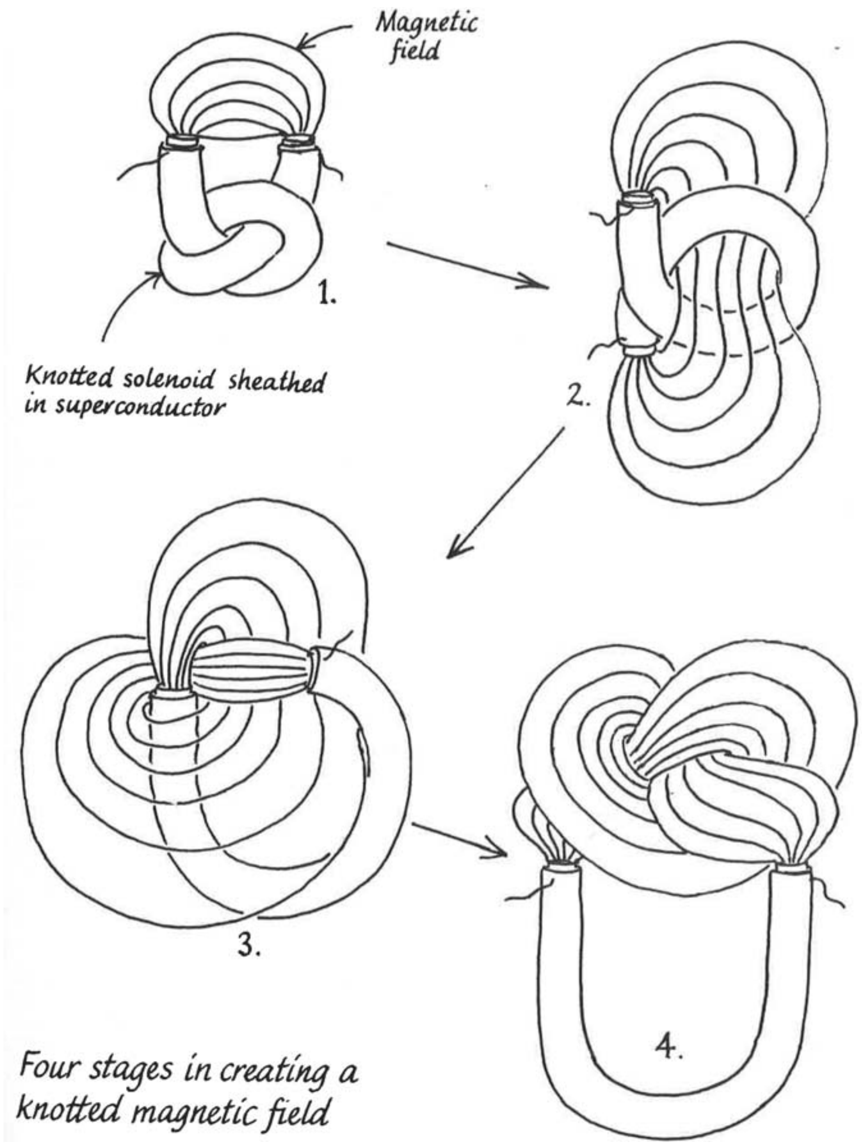
1 Problem: *To* B *or Knot to* B

In the "science humor" column by *Daedalus* (D.E.H. Jones) of 5 Oct. 1967 [1], the figure below was featured. Could such knotted configurations of magnetic field lines actually be generated?



For simplicity, suppose that the magnet is a long, thin permanent magnet of radius R small compared to its length, and magnetization density $\mathbf{M}$ of constant magnitude parallel to its (bendable) axis.

2 Solution

The magnetic fields $\mathbf{B}$ and $\mathbf{H}$ are related (in Gaussian units) by,

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \tag{1}$$

The field $\mathbf{B}$ obeys,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{2}$$

and hence,

$$\boldsymbol{\nabla} \cdot \mathbf{H} = 4\pi \boldsymbol{\nabla} \cdot \mathbf{M} \equiv 4\pi\rho_{\text{eff}}, \tag{3}$$

where $\rho_{\text{eff}} = \boldsymbol{\nabla} \cdot \mathbf{M}$ is an effective density of (fictitious) magnetic charge that can be said to generate field $\mathbf{H}$.

For magnets that are not bent too sharply (*i.e.*, whose local radius of curvature is large compared to the radius R of the magnet, transverse to its axis), we can ignore $\boldsymbol{\nabla} \cdot \mathbf{M} = \rho_{\text{eff}}$, and approximate that $\rho_{\text{eff}} \approx 0$ inside the magnet. However, on the ends of the magnet there exists effective surface charge $\sigma_{\text{eff}} = \mathbf{M} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit, outward normal to the (flat) surfaces at the two ends of the long magnet. Thus, we can suppose that the source of the $\mathbf{H}$ field are the two effective magnetic charges (effective magnetic poles) $\pm p = \pm MA$, where $A = \pi R^2$ is the area of the ends of the magnet. Hence, the $\mathbf{H}$ field is simply that of a magnetic dipole, no matter how "knotted" is the long magnet. **The lines of H can never be knotted for a long, thin permanent magnet.**

Of the four sketches by *Daedalus* on p. 1, only that for stage 1 is qualitatively correct for lines of $\mathbf{H}$ (which pattern of lines also holds for stages 2-4).

Outside the magnet, where $\mathbf{M} = 0$, the fields $\mathbf{B}$ and $\mathbf{H}$ are the same, and so the lines of $\mathbf{B}$ outside the magnet are also simply those of a magnetic dipole.

Inside the magnet, the field $\mathbf{H}$ is small compared to $\mathbf{M}$ except close to the ends of the magnet. To a good approximation, $\mathbf{B} = 4\pi\mathbf{M}$ inside the magnet, so lines of $\mathbf{B}$ inside the magnet are parallel to its axis.

Lines of $\mathbf{B}$ emerge from the end of the magnet with effective charge $p > 0$, and head over to the other end where they return back into the magnet.[1] Hence, if the magnet is "knotted", then the lines of $\mathbf{B}$ are also knotted, with the "knottiness" being inside, rather than outside, the magnet.

In the sense of the theory of topology, a knot is a closed curve, and the long, thin magnet does not form a knot unless its two ends are brought together. If this is done, then $\mathbf{H} \approx 0$ everywhere, and (to a first approximation) the magnetic field $\mathbf{B}$ in nonzero only inside the (knotted) magnet, and hence lines of $\mathbf{B}$ are knotted.

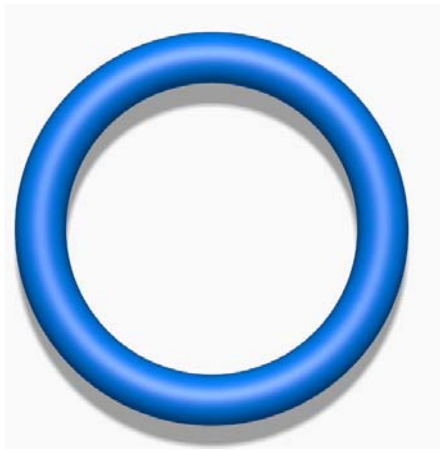
While lines of the magnetic field $\mathbf{B}$ can form nontrivial knots for a truly knotted magnet, the knotted field is very weak outside the magnet.

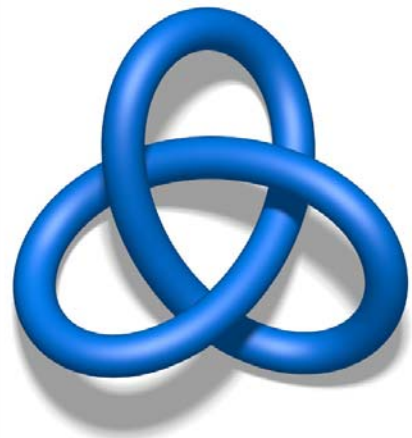
[1] The story is slightly more complicated in that some lines of $\mathbf{B}$ emerge from the cylindrical surface of the magnet along its entire length, to become the exterior field $\mathbf{B} = \mathbf{H}$ of the magnetic dipole formed by poles $\pm p$.

2.1 Magnetic Fields Lines Must "Break and Reconnect" as a Knot is Formed

Suppose we start with the long, thin magnet in the form of a circle, with the two ends of the magnet touching. Initially, lines of **B** form circles (which exist only inside the magnet, in the first approximation).

Then, we pull the ends of the magnet apart, bend the magnet into the simplest nontrivial (trefoil) knot, as shown on the right below, and rejoin the ends of the magnet.





Finally, lines of **B** again exist only inside the magnet (to a first approximation), and now they are (nontrivially) knotted. An originally circular line of **B** cannot be deformed into its final, trefoil configuration. In some sense, all of the lines of **B** have been "broken and reconnected" during the knotting process.

The concept of "breaking and reconnection" of magnetic field lines may have been introduced by Dungey [2], following suggestions by Giovanelli [3] and Hoyle [4] that interesting physics can occur near null points in the solar magnetic field.[2,3] However, unlike the physical magnet, a line of **B** cannot have a "free" end. The "breaking and reconnecting" must occur at a "null point", where the magnetic field is zero.[4] Illustrations from [2] of "breaking and reconnection" are shown on the next page.

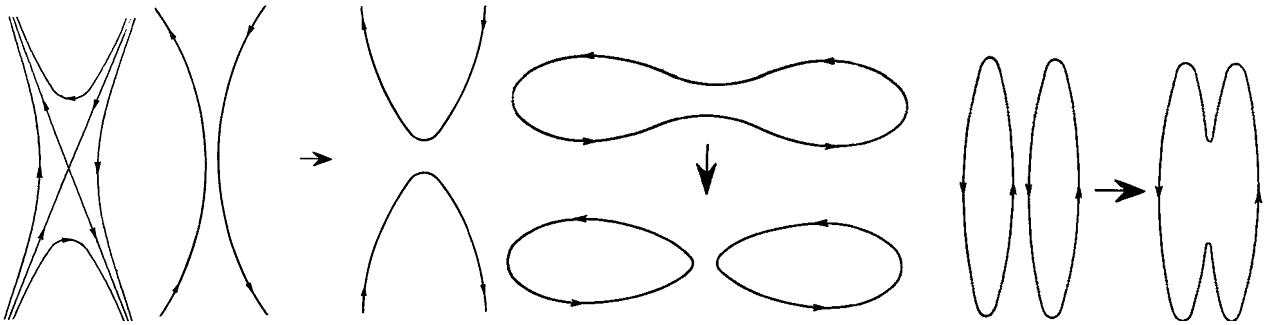
[2]For comments by the author on early astrophysical examples of "breaking and reconnection" of field lines, see [5].

[3]Slepian [6] has remarked that a field line is not a "mechanical" entity, and can be regarded has having an arbitrary number of breaks, whose ends are adjacent points along the field line.

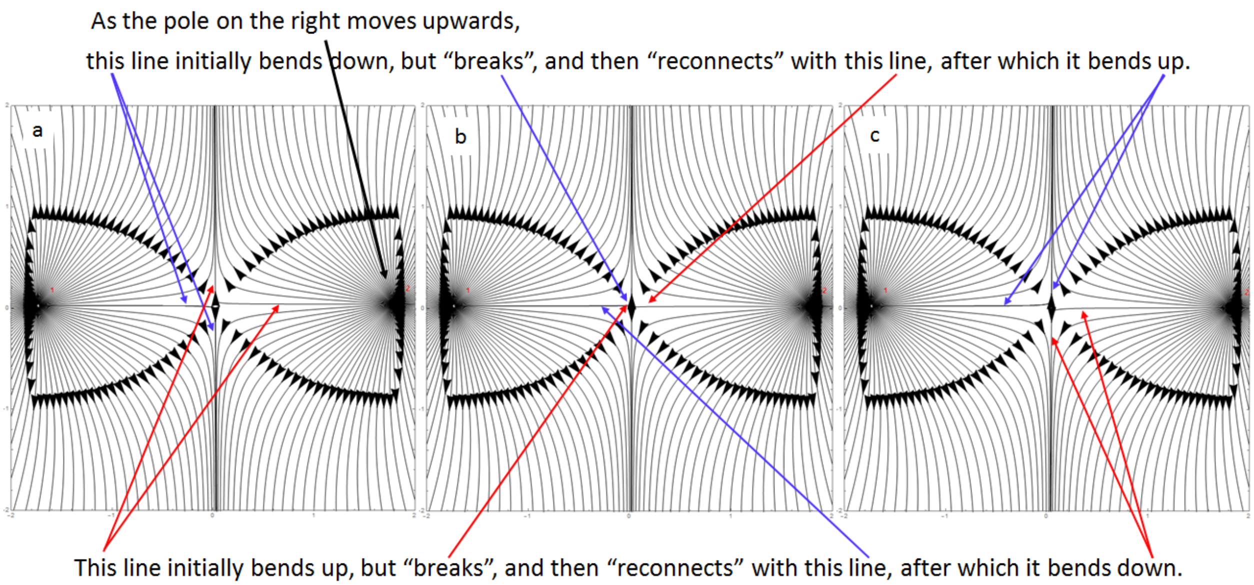
Slepian [6] was also one of the first to note that in complex current configurations one cannot be sure on the bases of analytic computations that a field line through a point returns to that point, or merely passes arbitrarily close to that point an infinite number of times. (If the latter holds, one should not infer that the magnetic field strength is infinite, as a field line should be counted only once in the concept that the field strength at a point in direction $\hat{\mathbf{n}}$ is proportional to the number of field lines crossing a small area, perpendicular to $\hat{\mathbf{n}}$, about the point). See also comments by Ulam [7, 8] on the "ergodic" character of magnetic field lines for the case of a steady loop of current together with a steady current along the axis of the loop.

Some people [8] seem to consider that field lines end at a null point, but this author considers that they do not, but rather all lines converging on the null point bend and diverge away from it.

[4]The magnetic field is zero at a null point. Such a point in a static electric field is a point of (unstable) equilibrium for electric charges, as considered by Maxwell in arts. 118-121 of his *Treatise*, and illustrated in his Figs. I-IV at the end of Vol. 1 [9].



A sketch of a magnetic field lines in the vicinity of a null point is shown in the figures below for a configuration of two like poles in which the separatrices happen to be at right angles. Three frames of an animation are shown, in which the pole to the right moves upwards, leading to "breaking" and "reconnection" of the field lines that point most directly from one pole to the other.



The above figure was generated via an interactive applet at
http://demonstrations.wolfram.com/ElectricFieldLinesDueToACollectionOfPointCharges/
One can drag the charges/poles around in applet, and the field lines will be recalculated in real time.

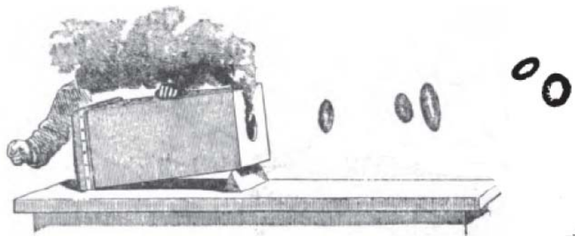
As the ends of the long, thin magnet are manipulated so as to form the trefoil knot, the claim is that all of the lines of **B** "break" and "reconnect".[5]

2.2 Vortices, Links and Knots (updated July 4, 2020)

One of the first discussions of a topologically nontrivial structure in physics was that by Helmholtz on hydrodynamic vortices (1858) [10]. Tait (1867) translated Helmholtz' paper, and conducted impressive lecture demonstrations of the generation of smoke rings/vortices;

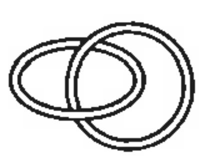
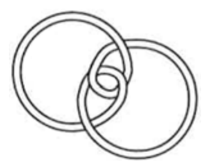
[5]This effect would occur in the example of Fig. 1, Vol. 1 of Maxwell's *Treatise* [9] if, say, the lower charge were displaced sideways, but seems to have gone unnoticed.

see p. 292 of [11]:[6]



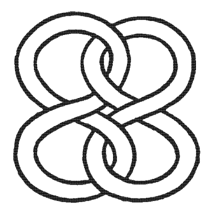
In 1867, p. 605 of [13], Gauss posthumously published a brief note, dated Jan. 22, 1833, on an integral to compute the linking number of a knot. See also [14, 15, 16].

This result was (independently?) deduced by Maxwell, in a letter to Tait of Nov. 13, 1867 [17], where Maxwell urged consideration of links and knots, examples of which he sketched as below.

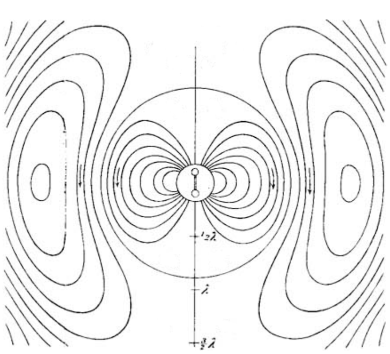
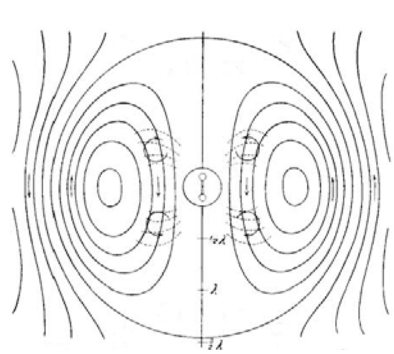
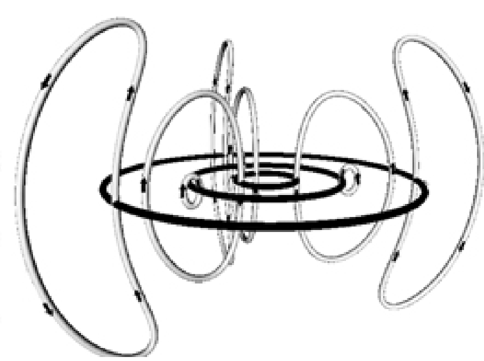



Subsequently, Tait made lengthy classifications of possible knots in electric or magnetic field lines [18], but did not much consider how such knots could be produced in the lab.

In Arts. 417-422 of [19], Maxwell discussed the linking-number integral, now attributing it to Gauss, and illustrated it with the linkage of two closed curves/field lines, each equivalent to a circle, as shown below.



Linked electric and magnetic field lines occur in electric dipole radiation, as discussed by Hertz [20], and illustrated in the figures below.[7]

Laser beams have been generated in which some field lines have the form of trefoil knot [22, 23]. For a review of the theory of knotted electromagnetic waves, see, for example, [24].

[6]Apparently, Tait's demonstration inspired W. Thomson (Lord Kelvin) to develop his theory of vortex atoms [12].

[7]The two figures on the left are from [20], and the one on the right is from [21].

Acknowledgment

The author thanks Ralph von Baltz for suggesting this problem.

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