Can an Electric-Current Density Be Replaced by an Equivalent Magnetization Density?

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1 Problem

It is often considered useful to regard a magnetization density $\mathbf{M}$ as the equivalent of a bulk electric-current density $\mathbf{J}$ plus a surface-current density $\mathbf{K}$, where (in SI units),

$$\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{M}, \qquad \mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}}, \tag{1}$$

and $\hat{\mathbf{n}}$ is the outward unit vector normal to the surface of the magnetized object.[1] Can we also suppose that any electric current density $\mathbf{J}$ is equivalent to a magnetization density $\mathbf{M}$?[2]

Is any electric charge density ρ equivalent to a polarization density $\mathbf{P}$ where $\rho = -\boldsymbol{\nabla} \cdot \mathbf{P}$?

2 Solution

In brief, the answer is NO, in the sense that while a mathematical equivalence can be exhibited, the equivalent magnetization has nonphysical properties, as discussed in sec. II.D.2 of [3] and noted in sec. 13.2.6 of [4]. Here, we elaborate.[3]

2.1 Maxwell's Argument (Mar. 3, 2022)

In Art. 637 of [6], Maxwell argued very concisely that: *It is impossible, by any arrangement of magnetized matter, to produce a system corresponding in all respects to an electric circuit, for the* (magnetic scalar) *potential of the magnetic system is single valued at every point of space, whereas that of the electric system is many-valued.*

2.2 Only a Steady Current Density Can Obey $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{M}$

If a current density obeys eq. (1), then,[4]

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0, \tag{2}$$

i.e., the current density is steady.

[1] An example of the utility of such equivalent current densities is given in sec. 3.1 of [1].

[2] One paper that suggests the answer to be YES is [2].

[3] That the answer is NO has been discussed by the author in a particular context in sec. 2.3 of [5].

[4] This contrasts with the fact that if magnetic charges do not exist (as is true so far as we know), then the magnetic field obeys $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ in any inertial frame, such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ for some vector potential $\mathbf{A}$ in any inertial frame.

Charge-current conservation tells us that,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{3}$$

where ρ is the electric charge density. Hence, if eq. (1) is to hold, the charge density must also be steady (time independent).

A charge or current density that is steady in one (inertial) frame of reference is not steady in any frame of reference that is in motion relative to the first frame. Hence, the relation (1) holds at most in a preferred reference frame, and is not generally consistent with the theory of special relativity.

We defer discussion of a relativistic form of an equivalence of charge and current densities to sec. 2.4.

2.3 Nonuniqueness of the Relation $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{M}$ for a Given Current Density $\mathbf{J}$

We might be content to analyze problems in which the current density $\mathbf{J}$ is steady only in a particular frame of reference. However, in that frame it remains that if $\mathbf{M}$ is an equivalent magnetization density which satisfies eq. (1), then the density $\mathbf{M}'$ does also, where,

$$\mathbf{M}' = \mathbf{M} + \boldsymbol{\nabla}\chi, \tag{4}$$

for any differentiable scalar function χ. That is, the magnetization $\mathbf{M}$ in relation (1) is like the vector potential $\mathbf{A}$ in the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, and so cannot be assigned a definite physical value, however useful it may be in computations involving the current density $\mathbf{J}$.[5]

2.3.1 Equivalent Magnetization and the Poincaré Gauge

Prescriptions for the equivalent magnetization $\mathbf{M}$ of a specified current density $\mathbf{J}$ can be deduced in analogy to the use of the so-called Poincaré gauge for the electromagnetic potentials V and $\mathbf{A}$.

We recall that in cases where the fields $\mathbf{E}$ and $\mathbf{B}$ are known, we can compute the potentials in the Poincaré gauge (see sec. 9A of [7] and [8, 9, 10]),[6]

$$V^{(\mathrm{P})}(\mathbf{r}, t) = -\mathbf{r} \cdot \int_0^1 du\, \mathbf{E}(u\mathbf{r}, t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{r}, t) = -\mathbf{r} \times \int_0^1 u\, du\, \mathbf{B}(u\mathbf{r}, t) \qquad \text{(Poincaré)}. \tag{5}$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.[7,8]

[5] Similarly, the transformation $\mathbf{P} \to \mathbf{P} + \boldsymbol{\nabla} \times \mathbf{F}$ for any field $\mathbf{F}$ leaves the polarization density $\mathbf{P}$ unchanged.

[6] The Poincaré gauge is also called the multipolar gauge [11].

[7] The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the "other side" from the origin.

[8] We transcribe Appendices C and D of [8] to verify that $\mathbf{E}$ and $\mathbf{B}$ indeed follow from the Poincaré

When $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ we can use the vector version of eq. (5) to give a form for the effective magnetization in the Poincaré gauge,

$$\mathbf{M}^{(\mathrm{P})}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 u \, du \, \mathbf{J}(u\mathbf{r}) \qquad \text{(Poincaré, steady currents).} \tag{8}$$

In contrast, the equivalent magnetization is written in the Appendix to [12], and in eq. (4.8) of [3], as,

$$\mathbf{M}^{(\mathrm{T})}(\mathbf{r}) = \mathbf{r} \times \int_1^\infty u \, du \, \mathbf{J}(u\mathbf{r}) \qquad \text{(Trammel, steady currents),} \tag{9}$$

which is valid, recalling eq. (7), for magnetic fields that fall off faster then $1/r$ for large r, as is the case for steady currents in a bounded volume. We call the form (9) the magnetization in the Trammel gauge.

2.4 Examples Where $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$

2.4.1 Uniform Ring of Current

Consider a ring (torus of major radius a and minor radius $b < a$, centered on the origin with $z = 0$ as its symmetry plane, that carries uniform current density $\mathbf{J} = J\,\hat{\boldsymbol{\phi}}$ in a spherical coordinate system (r, θ, ϕ). See the figure on the next page.

The equivalent magnetizations according to eqs. (8) and (9) are straightforwardly computed when the origin is at the center of the torus. Then, a ray of polar angle θ with $|\cos\theta| < \cos\theta_0 = b/a$ intersects the surface of the torus at radii r_1 and r_2 related by,

$$b^2 = a^2 + r_i^2 - 2ar_i \cos\alpha = a^2 + r_i^2 - 2ar_i \sin\theta, \qquad r_i = a\left(\sin\theta \pm \sqrt{\frac{b^2}{a^2} - \cos^2\theta}\right). \tag{10}$$

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potentials (5):

$$-\boldsymbol{\nabla} V^{(\mathrm{P})} - \frac{1}{c}\frac{\partial \mathbf{A}^{(\mathrm{P})}}{\partial t} = \int_0^1 du \left\{ \boldsymbol{\nabla}[\mathbf{r} \cdot \mathbf{E}(u\mathbf{r}, t)] + \mathbf{r} \times \frac{u}{c}\frac{\partial \mathbf{B}(u\mathbf{r}, t)}{\partial t} \right\} = \int_0^1 du \, \{\boldsymbol{\nabla}[\mathbf{r} \cdot \mathbf{E}(u\mathbf{r}, t)] - \mathbf{r} \times [\boldsymbol{\nabla} \times \mathbf{E}(u\mathbf{r}, t)]\}$$

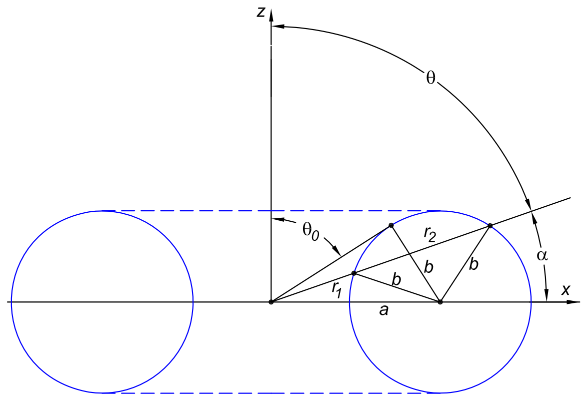
$$= \int_0^1 du \, \{(\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{E}(u\mathbf{r}, t) + [\mathbf{E}(u\mathbf{r}, t) \cdot \boldsymbol{\nabla}]\mathbf{r} + \mathbf{E}(u\mathbf{r}, t) \times (\boldsymbol{\nabla} \times \mathbf{r})\} = \int_0^1 du \left\{ u\frac{d(ux_i)}{du}\frac{\partial \mathbf{E}(u\mathbf{r}, t)}{\partial(ux_i)} + \mathbf{E}(u\mathbf{r}, t) \right\}$$

$$= \int_0^1 du \, \frac{d}{du} u\mathbf{E}(u\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t). \tag{6}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{P})} = -\int_0^1 u \, du \, \boldsymbol{\nabla} \times [\mathbf{r} \times \mathbf{B}(u\mathbf{r}, t)]$$

$$= -\int_0^1 u \, du \, \{\mathbf{r}[\boldsymbol{\nabla} \cdot \mathbf{B}(u\mathbf{r}, t)] - \mathbf{B}(u\mathbf{r}, t)[\boldsymbol{\nabla} \cdot \mathbf{r}] + [\mathbf{B}(u\mathbf{r}, t) \cdot \boldsymbol{\nabla}]\mathbf{r} - (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{B}(u\mathbf{r}, t)\}$$

$$= \int_0^1 u \, du \left\{ 2\mathbf{B}(u\mathbf{r}, t) + ux_i \frac{\partial \mathbf{B}(u\mathbf{r}, t)}{\partial(ux_i)} \right\} = \int_0^1 u \, du \left\{ \frac{1}{u}\frac{d}{du} u^2 \mathbf{B}(u\mathbf{r}, t) \right\} = \mathbf{B}(\mathbf{r}, t). \tag{7}$$



The equivalent magnetization of eq. (8) is nonzero along such a ray for all $r > r_1$, with,

$$\mathbf{M}^{(\mathrm{P})}(r_1 < r < r_2) = -\mathbf{r} \times J\,\hat{\boldsymbol{\phi}} \int_{r_1/r}^{1} u\,du = J\frac{r^2 - r_1^2}{2r}\,\hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}, \tag{11}$$

$$\mathbf{M}^{(\mathrm{P})}(r > r_2) = -\mathbf{r} \times J\,\hat{\boldsymbol{\phi}} \int_{r_1/r}^{r_2/r} u\,du = J\frac{r_2 - r_1^2}{2r}\,\hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}. \tag{12}$$

Note that the equivalent magnetization (12) is nonzero outside the torus, for $r_2 < r < \infty$, which reaffirms that this magnetization is nonphysical.

Similarly, the equivalent magnetization of eq. (9) is also nonzero along such a ray for all $r > r_1$, with,

$$\mathbf{M}^{(\mathrm{T})}(r < r_1) = \mathbf{r} \times J\,\hat{\boldsymbol{\phi}} \int_{r_1/r}^{r_2/r} u\,du = -J\frac{r_2^2 - r_1^2}{2r}\,\hat{\boldsymbol{\theta}} \qquad \text{(Trammel)}, \tag{13}$$

$$\mathbf{M}^{(\mathrm{T})}(r_1 < r < r_2) = \mathbf{r} \times J\,\hat{\boldsymbol{\phi}} \int_{1}^{r_2/r} u\,du = -J\frac{r_2^2 - r^2}{2r}\,\hat{\boldsymbol{\theta}} \qquad \text{(Trammel)}. \tag{14}$$

Again, the magnetization is nonzero outside the torus, but now for $r < r_1$ where the magnetization diverges as $r \to 0$, while the magnetization (14) inside the torus has the opposite sign to that of eq. (11). This illustrates that the equivalent magnetization is not gauge invariant.

Of course, the magnetizations (11)-(14) satisfy $\mathbf{J} = \boldsymbol{\nabla} \times \mathbf{M} = J\,\hat{\boldsymbol{\phi}}$ inside the torus, and $\mathbf{J} = 0$ outside, as is readily verified by direct differentiation, noting that $J_\phi = (1/r)\partial(rM_\theta)/\partial r$.

2.4.2 Rotating, Uniformly Charged Sphere

For a sphere of radius a with uniform electric charge density ρ, centered on the origin and rotating with angular velocity ω about the z-axis, the current density is $\mathbf{J}(r < a) = \rho\omega r \sin\theta\,\hat{\boldsymbol{\phi}}$ in a spherical coordinate system (r, θ, ϕ).

The equivalent magnetization of eq. (8) is nonzero throughout all space, with,

$$\mathbf{M}^{(\mathrm{P})}(r < a) = -\mathbf{r} \times \int_0^1 u\,du\,\rho\omega\,(ur)\sin\theta\,\hat{\boldsymbol{\phi}} = \frac{\rho\omega r^2 \sin\theta}{3}\,\hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}, \tag{15}$$

$$\mathbf{M}^{(\mathrm{P})}(r > a) = -\mathbf{r} \times \int_0^{a/r} u\,du\,\rho\omega\,(ur)\sin\theta\,\hat{\boldsymbol{\phi}} = \frac{\rho\omega a^3 \sin\theta}{3r}\,\hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}, \tag{16}$$

which vanishes at the origin.

The magnetization of eq. (9) is nonzero only inside the rotating sphere, with,

$$\mathbf{M}^{(\mathrm{T})}(r<a)=\mathbf{r}\times\int_1^{a/r}u\,du\,\rho\omega\,(ur)\sin\theta\,\hat{\boldsymbol{\phi}}=-\rho\omega\sin\theta\frac{a^3-r^3}{3r}\,\hat{\boldsymbol{\theta}}\qquad\text{(Trammel)},\qquad(17)$$

which vanishes at the surface of the sphere, and diverges at the origin.

While the Poincaré-gauge equivalent magnetization (15)-(16) is well behaved at the origin, it is nonzero outside the physical sphere. In contrast, the Trammel-gauge magnetization (17) is zero outside the physical sphere, but diverges at the origin. Both equivalent magnetizations are "mathematical fictions" without direct "physical reality".

2.5 Relativistic Equivalents for Charge and Current Densities

We can devise a relativistic generalization of eq. (1) by introducing the four vectors and tensors,

$$J^\alpha=(c\rho,\mathbf{J}),\qquad\partial_\beta=\left(\frac{\partial}{\partial ct},\boldsymbol{\nabla}\right),\qquad M^{\alpha\beta}=\begin{pmatrix}0&cP_x&cP_y&cP_x\\-cP_x&0&-M_x&M_y\\-cP_y&M_z&0&-M_x\\-cP_z&-M_y&M_x&0\end{pmatrix},\qquad(18)$$

where $\mathbf{P}$ is the density of electric-dipole moments (polarization density) and c is the speed of light in vacuum. Then, in the Einstein notation that $a_\alpha b^\alpha=\sum_{\mu=0}^3 a_\mu b^\mu$, we could write,

$$J^\alpha=\partial_\beta M^{\beta\alpha}=\left(-c\boldsymbol{\nabla}\cdot\mathbf{P},\frac{\partial\mathbf{P}}{\partial t}+\boldsymbol{\nabla}\times\mathbf{M}\right),\qquad(19)$$

with the implication that the charge and current densities might be represented by equivalent polarization and magnetization densities $\mathbf{P}$ and $\mathbf{M}$, where,

$$\rho=-\boldsymbol{\nabla}\cdot\mathbf{P},\qquad\mathbf{J}=\frac{\partial\mathbf{P}}{\partial t}+\boldsymbol{\nabla}\times\mathbf{M}.\qquad(20)$$

2.5.1 Equivalent Polarization Density P in the Poincaré and Trammel Gauges

We supplement the expression (8) for the magnetization in the Poincaré gauge by the relation,

$$\mathbf{P}^{(\mathrm{P})}(\mathbf{r})=-\mathbf{r}\int_0^1 u^2\,du\,\rho(u\mathbf{r})\qquad\text{(Poincaré)},\qquad(21)$$

for the polarization. Then,

$$\begin{aligned}-\boldsymbol{\nabla}\cdot\mathbf{P}^{(\mathrm{P})}&=\int_0^1 u^2\,du\,[\rho(u\mathbf{r})\boldsymbol{\nabla}\cdot\mathbf{r}+(\mathbf{r}\cdot\boldsymbol{\nabla})\rho(u\mathbf{r})]=\int_0^1 u^2\,du\left[3\rho(u\mathbf{r})+u\frac{d(ux_i)}{du}\frac{\partial\rho(u\mathbf{r})}{\partial x_i}\right]\\&=\int_0^1 u^2\,du\left\{\frac{1}{u^2}\frac{d}{du}\left[u^3\rho(u\mathbf{r})\right]\right\}=\rho(\mathbf{r}),\end{aligned}\qquad(22)$$

such that the first of eq. (20) is satisfied. Also,

$$\frac{\partial \mathbf{P}^{(\mathrm{P})}}{\partial t} = -\int_0^1 u^2\,du\,\mathbf{r}\frac{\partial \rho(u\mathbf{r})}{\partial t} = \int_0^1 u^2\,du\,\mathbf{r}\frac{\partial J_i(u\mathbf{r})}{\partial(ux_i)} = \int_0^1 u\,du\,\mathbf{r}[\boldsymbol{\nabla}\cdot\mathbf{J}(u\mathbf{r})], \tag{23}$$

which cancels the term $-\int_0^1 u\,du\,\mathbf{r}[\boldsymbol{\nabla}\cdot\mathbf{J}(u\mathbf{r})]$ that arises when computing $\boldsymbol{\nabla}\times\mathbf{M}^{(\mathrm{P})}$ for a nonsteady current density $\mathbf{J}$ (recalling eq. (7) for $\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{P})}$ in terms of the magnetic field $\mathbf{B}$).

That is, the expression (8) (and also (9)) for the equivalent magnetization density $\mathbf{M}$ is also valid when $\boldsymbol{\nabla}\cdot\mathbf{J}\neq 0$, in which case $\boldsymbol{\nabla}\times\mathbf{M} = \mathbf{J} - \partial\mathbf{P}/\partial t$.

Similarly, the polarization density in the Trammel gauge is,

$$\mathbf{P}^{(\mathrm{T})}(\mathbf{r}) = \mathbf{r}\int_1^\infty u^2\,du\,\rho(u\mathbf{r}) \qquad \text{(Trammel)}, \tag{24}$$

which as before only applies when the fields fall off sufficiently quickly at large distances.

2.6 Examples Where $\boldsymbol{\nabla}\cdot\mathbf{J}\neq 0$

2.6.1 Uniform Sphere of Charge with $v \ll c$

We consider a uniform sphere of charge density ρ, of radius a and centered on the origin in its rest frame, in the "lab" frame where the charge has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ with $v \ll c$, such that the volume or charge is essentially spherical. At the time when the charge is centered on the origin in the lab frame, the current and current densities are, with neglect of terms of order v^2/c^2,

$$\rho(r<a) = \rho, \qquad \mathbf{J}(r<a) = \rho v\,\hat{\mathbf{z}}, \tag{25}$$

and zero for $r > a$.

In the Poincaré gauge, the equivalent polarization density is,

$$\mathbf{P}^{(\mathrm{P})}(r<a) = -\mathbf{r}\int_0^1 u^2\,du\,\rho = -\frac{\rho r}{3}\,\hat{\mathbf{r}} \qquad \text{(Poincaré)}, \tag{26}$$

$$\mathbf{P}^{(\mathrm{P})}(r>a) = -\mathbf{r}\int_0^{a/r} u^2\,du\,\rho = -\frac{\rho a^3}{3r^2}\,\hat{\mathbf{r}} \qquad \text{(Poincaré)}, \tag{27}$$

for which $-\boldsymbol{\nabla}\cdot\mathbf{P} = -(1/r^2)d(r^2P_r)/dr = \rho$ for $r < a$ and zero for $r > a$. Similarly, the equivalent magnetization is,

$$\mathbf{M}^{(\mathrm{P})}(r<a) = -\mathbf{r}\times\int_0^1 u\,du\,\rho v\,\hat{\mathbf{z}} = -\frac{\rho r v\sin\theta}{2}\,\hat{\boldsymbol{\phi}} \qquad \text{(Poincaré)}, \tag{28}$$

$$\mathbf{M}^{(\mathrm{P})}(r>a) = -\mathbf{r}\times\int_0^{a/r} u\,du\,\rho v\,\hat{\mathbf{z}} = -\frac{\rho a^2 v\sin\theta}{2r}\,\hat{\boldsymbol{\phi}} \qquad \text{(Poincaré)}. \tag{29}$$

Both the equivalent polarization and magnetization are nonzero throughout all space.

In the Trammel gauge, the equivalent polarization density is nonzero only for $r < a$,

$$\mathbf{P}^{(\mathrm{T})}(r < a) = \mathbf{r} \int_1^{a/r} u^2\, du\, \rho = \frac{\rho(a^3 - r^3)}{3r^2}\, \hat{\mathbf{r}} \qquad \text{(Trammel)}, \tag{30}$$

which diverges at the origin. Similarly, the equivalent magnetization is,

$$\mathbf{M}^{(\mathrm{T})}(r < a) = \mathbf{r} \times \int_1^{a/r} u\, du\, \rho v\, \hat{\mathbf{z}} = -\frac{\rho v(a^2 - r^2)\sin\theta}{2r}\, \hat{\boldsymbol{\phi}} \qquad \text{(Trammel)}, \tag{31}$$

which also diverges at the origin.

2.6.2 Current Loop with Velocity along Its Axis

We consider again the current loop of sec. 2.3.1, now with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, where $v \ll c$, at the moment when the loop is centered on the origin. With neglect of effects of order v^2/c^2, the shape of the loop is the same as in its rest frame, and the loop can be regarded as electrically neutral in that frame.[9] That is, in the rest frame the charge density is $\rho^\star = 0$ (at order v/c) and the current density is $\mathbf{J}^\star = J\,\hat{\boldsymbol{\phi}}$. Then, in the lab frame the charge density is $\rho = \gamma(\rho^\star + \mathbf{J}^\star\cdot\mathbf{v}/c^2) = \gamma\rho^\star = 0$, and the current density is $\mathbf{J} = \mathbf{J}^\star + (\gamma-1)(\mathbf{J}^\star\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - \gamma\rho^\star\mathbf{v} = \mathbf{J}^\star$, where $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$. Since the charge and current densities are the same in the lab frame as in the rest frame, the equivalent magnetization $\mathbf{M}$ in the lab frame is the same as that found in sec. 2.3.1 (and the equivalent polarization density $\mathbf{P}$ is zero).

2.6.3 Current Loop with Velocity in Its Symmetry Plane

We consider again the current loop of sec. 2.3.1, now with velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, where $v \ll c$, at the moment when the loop is centered on the origin. With neglect of effects of order v^2/c^2, the shape of the loop is the same as in its rest frame, and the loop can be regarded as electrically neutral in that frame. That is, in the rest frame the charge density is $\rho^\star = 0$ and the current density is $\mathbf{J}^\star = J\,\hat{\boldsymbol{\phi}} = J(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})$. Then, in the lab frame the charge density is,

$$\rho = \gamma\left(\rho^\star + \frac{\mathbf{J}^\star\cdot\mathbf{v}}{c^2}\right) \approx \frac{Jv\cos\phi}{c^2}\,, \tag{32}$$

and the current density is,

$$\mathbf{J} = \mathbf{J}^\star + (\gamma - 1)(\mathbf{J}^\star\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} - \gamma\rho^\star\mathbf{v} \approx \mathbf{J}^\star = J\,\hat{\boldsymbol{\phi}}. \tag{33}$$

Since the current density is the same in the lab frame as in the rest frame, the equivalent magnetization $\mathbf{M}$ in the lab frame is the same as that found in sec. 2.3. However, the nonzero charge density (32) in the lab frame implies that there is a nonzero equivalent polarization density $\mathbf{P}$ as well.

[9] At order v_e^2/c^2, where v_e is the drift velocity of the conduction electrons, the current loop would have a small bulk charge density in its rest frame, whose resulting electric field would cancel the $\mathbf{J}\times\mathbf{B}$ force on the current [13].

In the Poincaré gauge, the lab-frame equivalent polarization density is,

$$\mathbf{P}^{(\mathrm{P})}(r_1 < r < r_2) = -\mathbf{r}\int_{r_1/r}^{1} u^2\, du\, \rho = -\frac{\rho(r^3 - r_1^3)}{3r^2}\,\hat{\mathbf{r}} = -\frac{Jv(r^3 - r_1^3)\cos\phi}{3c^2r^2}\,\hat{\mathbf{r}} \qquad \text{(Poincaré)}, (34)$$

$$\mathbf{P}^{(\mathrm{P})}(r > r_2) = -\mathbf{r}\int_{r_1/r}^{r_2/r} u^2\, du\, \rho = -\frac{\rho(r_2^3 - r_1^3)}{3r^2}\,\hat{\mathbf{r}} = -\frac{Jv(r_2^3 - r_1^3)\cos\phi}{3c^2r^2}\,\hat{\mathbf{r}} \qquad \text{(Poincaré)}, (35)$$

which (like the magnetization) is nonzero outside the current loop.

In the Trammel gauge, the lab-frame equivalent polarization density is,

$$\mathbf{P}^{(\mathrm{T})}(r < r_1) = \mathbf{r}\int_{r_1/r}^{r_2/r} u^2\, du\, \rho = \frac{\rho(r_2^3 - r_1^3)}{3r^2}\,\hat{\mathbf{r}} = \frac{Jv(r_2^3 - r_1^3)\cos\phi}{3c^2r^2}\,\hat{\mathbf{r}} \qquad \text{(Trammel)}, (36)$$

$$\mathbf{P}^{(\mathrm{T})}(r_1 < r < r_2) = \mathbf{r}\int_{1}^{r_2/r} u^2\, du\, \rho = \frac{\rho(r_2^3 - r^3)}{3r^2}\,\hat{\mathbf{r}} = \frac{Jv(r_2^3 - r_1^3)\cos\phi}{3c^2r^2}\,\hat{\mathbf{r}} \qquad \text{(Trammel)}, (37)$$

which (like the magnetization) diverges at the origin.

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