What is i^i?

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This amusing problem is posed by R.C. Henry at http://henry.pha.jhu.edu/rch.html

Henry's solution invokes the facts that $(a^b)^c = a^{bc}$ and that one can write $i = e^{(4n+1)i\pi/2}$, for any real integer n, to find the infinite set of answers,

$$i^i = (e^{(4n+1)i\pi/2})^i = e^{-(4n+1)\pi/2}. \tag{1}$$

Henry favors setting $n = 0, \quad \Rightarrow \quad i^i = e^{-\pi/2} \approx 0.208.$

Following Henry's advice to try the problem before looking at his solution, I proceeded differently,

$$i^i = x = e^{\ln x} = e^{-(4n+1)\pi/2}, \tag{2}$$

as in eq. (1), using,

$$\ln x = i \ln i = i \ln(1\, e^{i\pi/2}) = i[0 + (4n+1)i\pi/2] = -(4n+1)\pi/2, \tag{3}$$

which follows from the definition of the logarithm of a complex number,

$$\ln z = \ln r + i(\phi + 2n\pi), \qquad \text{where} \qquad z = r\, e^{i\phi}. \tag{4}$$