

Index of Refraction of a Moving Medium

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1 Problem

A linear, isotropic medium at rest can be characterized by a (frequency-dependent) index of refraction $n(\omega)$. What is the index of refraction in a frame in which this medium has velocity \mathbf{v} ?

2 Solution

This problem is implicit in early experiments to search for effects of the supposed æther in moving media, such as that of Arago (1810) [1] and Fizeau (1851) [2, 3, 4], and was addressed by Lorentz in 1903 [5].¹ Discussion in the context of special relativity was first given by Laub [7], and then more clearly by Laue [8], both in 1907.² See also sec. 36 of Pauli's review [10]. Perhaps because these works were written in German, awareness of them seems to have faded in the English literature, where discussions such as [11]-[14] make no mention of them.

We write a 4-vector as $a_\mu = (a_0, \mathbf{a})$ and the square of its invariant length as $a_\mu a^\mu = a_0^2 - \mathbf{a}^2$. In particular, the position 4-vector is $x_\mu = (ct, \mathbf{x})$, where c is the speed of light in vacuum.

The Lorentz transformation of a 4-vector a_μ from one inertial frame to another with velocity \mathbf{v} with respect to the first can be written as,

$$a'_0 = \gamma(a_0 - a_{\parallel}v/c), \quad (1)$$

$$\mathbf{a}'_{\parallel} = \gamma(\mathbf{a}_{\parallel} - a_0\mathbf{v}/c), \quad (2)$$

$$\mathbf{a}'_{\perp} = \mathbf{a}_{\perp}, \quad (3)$$

where,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}, \quad \mathbf{a}_{\parallel} = (\mathbf{a} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \quad (4)$$

and the 4-vector has components (a_0, \mathbf{a}) and (a'_0, \mathbf{a}') in the first and second frames.

We recall that the 3-velocity \mathbf{u} of a particle obeys $u < c$ and can be embedded in the 4-velocity,

$$u_\mu = \gamma_u(c, \mathbf{u}), \quad \text{where} \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}. \quad (5)$$

The Lorentz transformation (1)-(3) of the particle 4-velocity is,

$$\gamma_{u'} = \gamma\gamma_u(1 - \mathbf{u} \cdot \mathbf{v}/c^2), \quad (6)$$

¹Lorentz discussed his eponymous transformations with respect to a stationary æther), prior to Einstein's derivation thereof (1905) [6] (without an æther).

²For discussion of how after 1907 Einstein sometimes stated that an awareness of Fizeau's experiment was a significant motivation for his development of the theory of relativity, see [9].

$$\mathbf{u}'_{\parallel} = \frac{\gamma\gamma_u(\mathbf{u}_{\parallel} - \mathbf{v})}{\gamma_{u'}} = \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u} \cdot \mathbf{v}/c^2} = \frac{(\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \mathbf{v}}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}, \quad (7)$$

$$\mathbf{u}'_{\perp} = \mathbf{u}_{\perp} = \mathbf{u} - (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}. \quad (8)$$

Equations (7) and (8) can be combined to give,

$$\mathbf{u}' = \frac{\mathbf{u} - \mathbf{v} - \mathbf{u}_{\perp}(\mathbf{u} \cdot \mathbf{v})/c^2}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}, \quad (9)$$

which reverts to the Galilean transformation, $\mathbf{u}' = \mathbf{u} - \mathbf{v}$, of particle velocity in the low-velocity limit.

2.1 Lorentz Transformation of Phase Velocity

As will be confirmed below, phase velocity is not the spatial part of a 4-vector, and does not transform according to eq. (9). However, we can still display an explicit form for the Lorentz transformation of phase velocity via considerations of the (Lorentz invariant) phase of a wave.

In particular, we can associate the plane wave, $\cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$, with a wave 4-vector $k_{\mu} = (\omega/c, \mathbf{k})$ such that the phase ϕ of the wave is a Lorentz scalar,

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t = -k_{\mu}x^{\mu}. \quad (10)$$

Surfaces of constant phase propagate with phase velocity,

$$\mathbf{u}_p(\omega, \mathbf{k}) = \frac{\omega}{k^2}\mathbf{k} = \mathbf{u}_{p\parallel} + \mathbf{u}_{p\perp} = u_{p\parallel}\hat{\mathbf{v}} + u_{p\perp}\hat{\mathbf{k}}_{\perp}, \quad (11)$$

where $\mathbf{u}_{p\parallel} = \mathbf{u} \cdot \hat{\mathbf{v}}$, and $\hat{\mathbf{k}}_{\perp}$ is a unit vector in the direction of $\mathbf{u}_{p\perp} = \mathbf{u}_p - \mathbf{u}_{p\parallel}$ (and not the component perpendicular to \mathbf{v} of the unit vector $\hat{\mathbf{k}} = \hat{\mathbf{u}}_p$). If we write,

$$\mathbf{k} = k_{\parallel}\hat{\mathbf{v}} + k_{\perp}\hat{\mathbf{k}}_{\perp}, \quad (12)$$

and note that $\omega = k u_p$, we also have,

$$k_{\parallel} = \frac{\omega u_{p\parallel}}{u_p^2}, \quad k_{\perp} = \frac{\omega u_{p\perp}}{u_p^2}. \quad (13)$$

In a frame that moves with velocity \mathbf{v} with respect to the original frame the phase velocity is,

$$\mathbf{u}'_p = \frac{\omega'}{k'^2}\mathbf{k}' = \frac{\omega'}{k'}\hat{\mathbf{u}}'_p = \mathbf{u}'_{p\parallel} + \mathbf{u}'_{p\perp}. \quad (14)$$

The Lorentz transformation (1)-(3) of the wave 4-vector k_{μ} is,

$$\omega' = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) = \gamma(\omega - k_{\parallel}v) = \frac{\gamma\omega}{u_p^2}(u_p^2 - u_{p\parallel}v), \quad (15)$$

$$\mathbf{k}'_{\parallel} = \gamma(\mathbf{k}_{\parallel} - \omega\mathbf{v}/c^2) = \gamma\hat{\mathbf{v}}(k_{\parallel} - \omega v/c^2) = \frac{\gamma\omega}{c^2 u_p^2}\hat{\mathbf{v}}(c^2 u_{p\parallel} - u_p^2 v), \quad (16)$$

$$\mathbf{k}'_{\perp} = \mathbf{k}_{\perp} = k_{\perp}\hat{\mathbf{k}}_{\perp} = \frac{\omega u_{p\perp}}{u_p^2}\hat{\mathbf{k}}_{\perp}. \quad (17)$$

Combining eqs. (16) and (17) we find,

$$k'^2 = \frac{\omega^2}{c^4 u_p^4} [c^4 u_{p\perp}^2 + \gamma^2 (c^2 u_{p\parallel} - u_p^2 v)^2], \quad (18)$$

and hence the Lorentz transformation of phase velocity is,

$$u'_p = \frac{\omega'}{k'} = \frac{\gamma c^2 (u_p^2 - u_{p\parallel} v)}{\sqrt{c^4 u_{p\perp}^2 + \gamma^2 (c^2 u_{p\parallel} - u_p^2 v)^2}} = \frac{u_p - v u_{p\parallel}/u_p}{\sqrt{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p v/c^2)^2}}, \quad (19)$$

and the direction of the wavefront is,

$$\hat{\mathbf{u}}'_p = \hat{\mathbf{k}}' = \frac{\mathbf{k}'_{\parallel} + \mathbf{k}'_{\perp}}{k'} = \frac{\gamma (c^2 u_{p\parallel} - u_p^2 v) \hat{\mathbf{v}} + c^2 u_{p\perp} \hat{\mathbf{k}}_{\perp}}{\sqrt{c^4 u_{p\perp}^2 + \gamma^2 (c^2 u_{p\parallel} - u_p^2 v)^2}} = \frac{(u_{p\parallel}/u_p - u_p v/c^2) \hat{\mathbf{v}} + (u_{p\perp}/\gamma u_p) \hat{\mathbf{k}}_{\perp}}{\sqrt{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p v/c^2)^2}}. \quad (20)$$

The components of the transformed phase velocity are,

$$\mathbf{u}'_{p\parallel} = u'_p \frac{\mathbf{k}'_{\parallel}}{k'} = \frac{\gamma^2 c^2 (u_p^2 - u_{p\parallel} v) (c^2 u_{p\parallel} - u_p^2 v)}{c^4 u_{p\perp}^2 + \gamma^2 (c^2 u_{p\parallel} - u_p^2 v)^2} \hat{\mathbf{v}} = \frac{(u_p - v u_{p\parallel}/u_p) (u_{p\parallel}/u_p - u_p v/c^2)}{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p v/c^2)^2} \hat{\mathbf{v}}, \quad (21)$$

$$\mathbf{u}'_{p\perp} = u'_p \frac{\mathbf{k}'_{\perp}}{k'} = \frac{\gamma c^4 (u_p^2 - u_{p\parallel} v) u_{p\perp}}{c^4 u_{p\perp}^2 + \gamma^2 (c^2 u_{p\parallel} - u_p^2 v)^2} \hat{\mathbf{k}}_{\perp} = \frac{(u_p - v u_{p\parallel}/u_p) u_{p\perp}/\gamma u_p}{u_{p\perp}^2/\gamma^2 u_p^2 + (u_{p\parallel}/u_p - u_p v/c^2)^2} \hat{\mathbf{k}}_{\perp}. \quad (22)$$

For the special case that $u_p = c$, eq. (15) becomes $\omega' = \gamma \omega (1 - u_{p\parallel} v/c^2)$, and eq. (18) becomes (after some algebra) $k'^2 = \gamma^2 \omega^2 (1 - u_{p\parallel} v/c^2)^2 / c^2 = \omega'^2 / c^2$. That is, if the phase velocity is c in one inertial frame it is also c in any other inertial frame.

The comments in the rest of this section suppose that the phase velocity differs from c in the rest frame of the medium.

From eq. (19) we see that the phase velocity in the moving medium depends on the direction of the wave, so that the moving medium appears to be anisotropic even if the medium at rest is isotropic.

We defer discussion of dispersion in the moving medium to sec. 2.2.

If the boost velocity \mathbf{v} is parallel to the phase velocity \mathbf{u}_p , then from eqs. (15) and (16), or from eq. (19), we find that,

$$\mathbf{u}'_p = \frac{\mathbf{u}_p - \mathbf{v}}{1 - \mathbf{u}_p \cdot \mathbf{v}/c^2} \quad (\mathbf{u}_p \parallel \mathbf{v}), \quad (23)$$

so that the Lorentz transformation of phase velocity is the same as that for particle velocity, eq. (9), in this special case.

However, since the phase speed ω/k can have any value less than, equal to, or even greater than c ,³ the wave 4-vector k_{μ} can be spacelike, lightlike or timelike, respectively. This permits possibly surprising scenarios such as $\mathbf{u}_p = 2c \hat{\mathbf{x}}$, $\mathbf{v} = c \hat{\mathbf{x}}/2$, for which the transformed wave vector \mathbf{k}' vanishes, the transformed waveform is the standing wave $\cos(\sqrt{3/4} \omega t')$, and the transformed phase velocity is formally infinite.

³A well-known case with phase speed greater than c is a waveguide. See, for example, sec. 8.3 of [16].

2.2 Propagation of Light in a Moving Medium

In the case of propagation of light in a medium with an index of refraction different from 1, the rest frame of that medium is a preferred frame. In this frame, the phase velocity (11) can be written as,

$$\mathbf{u}_p = \frac{\omega}{k} \hat{\mathbf{u}}_p = \frac{c}{n} \hat{\mathbf{u}}_p, \quad (24)$$

i.e., $u_p = c/n$, such that the phase velocity (19) in a frame where the medium has velocity \mathbf{v} at angle α to the wave vector \mathbf{k} (in the rest frame of the medium) can be written as,⁴

$$\mathbf{u}'_p = \frac{1 - n\beta \cos \alpha}{\sqrt{n^2(1 - \cos^2 \alpha)(1 - \beta^2) + (n \cos \alpha - \beta)^2}} c \hat{\mathbf{u}}'_p \equiv \frac{c}{n'} \hat{\mathbf{u}}'_p, \quad (25)$$

noting that $u_{p\parallel}/u_p = \cos \alpha$. Then, the index of refraction n' in the moving medium is given by,

$$\begin{aligned} n' &= \frac{\sqrt{n^2(1 - \cos^2 \alpha)(1 - \beta^2) + (n \cos \alpha - \beta)^2}}{1 - n\beta \cos \alpha} \\ &= \frac{\sqrt{n^2(1 + \beta^2 \cos^2 \alpha) - 2n\beta \cos \alpha - (n^2 - 1)\beta^2}}{1 - n\beta \cos \alpha} = \frac{k'c}{\omega'}, \end{aligned} \quad (26)$$

which is a function of the angle α and index of refraction n in the rest frame of the medium.⁵ If the medium at rest is nondispersive, the index n is independent of the angular frequency ω , and likewise the index n' is independent of ω' , so the moving medium also is nondispersive (but anisotropic).⁶

Only for the special case that the phase velocity is c are the waves both nondispersive and isotropic in all inertial frames.

For low velocities, $v \ll c$, the phase velocity u'_p in the frame where the medium has velocity \mathbf{v} follows from eq. (25) as,

$$u'_p \approx \frac{c}{n} - v \cos \alpha \left(1 - \frac{1}{n^2}\right). \quad (27)$$

This expression was predicted by Fresnel (1818) [17], via an argument about partial dragging of the æther by moving media, inspired in part by the null result of an ether-drift experiment by Arago (1810) [1] that added a prism to a telescope.⁷ The expression (27) was confirmed

⁴Equation (25) was first given by Laue (1907) [8], where quantities in the rest frame of the medium were labeled with a $'$, and our $\cos \alpha$ is Laue's $-\cos \vartheta'$. See also eq. (311c) of Pauli's 1921 review [10].

⁵The naïve assumption that the index of refraction of a moving medium is isotropic if it is so in its rest frame can lead to spurious deductions, as in [13].

⁶This conclusion is reached in [14], where a much less compact expression for the index (26) in the moving medium is given.

⁷The intricate story of thought experiments on stellar aberration that influenced Fresnel is reviewed in Pederson [18]. See also [19, 20]. For additional commentary on Arago's experiment, see [21]. The thought experiments with water-filled telescopes were first realized, with the expected negative results, around 1870 [22, 23].

experimentally, for $\cos \alpha = -1$, by Fizeau (1851) [2, 3, 4].^{8,9} Lorentz (1903) deduced eq. (27) for $\cos \alpha = -1$ in sec. 60 of [5], using his electron theory with a stationary æther that slightly preceded Einstein’s (1905) theory of special relativity [6]. It is noteworthy that eq. (27) is a “special relativistic” effect observable for $v \ll c$ that is not anticipated by use of Galilean relativity (in contrast to stellar aberration, discussed by Einstein in sec. 7 of [6]).

Appendix: Lorentz Transformation of Group Velocity

This Appendix follows [28]. See also [29, 30].

A suitable 4-vector generalization of the 3-dimensional group velocity, $\mathbf{u}_g = \partial\omega/\partial\mathbf{k}$, is based on rewriting the dispersion relation $\omega = \omega(\mathbf{k})$ as,

$$F(k_\mu) = F(\omega/c, \mathbf{k}) = 0, \quad (28)$$

and then taking the 4-vector gradient,

$$\left. \frac{\partial F}{\partial k_\mu} \right|_{F=0} = \left(c \frac{\partial F}{\partial \omega}, \frac{\partial F}{\partial \mathbf{k}} \right)_{F=0} = \left. \frac{\partial F}{\partial \omega} \right|_{F=0} \left(c, \frac{\partial \omega}{\partial \mathbf{k}} \right) = \left. \frac{\partial F}{\partial \omega} \right|_{F=0} (c, \mathbf{u}_g). \quad (29)$$

The invariant length of this gradient is,

$$\sqrt{\left. \frac{\partial F}{\partial k_\mu} \frac{\partial F}{\partial k^\mu} \right|_{F=0}} = c \left. \frac{\partial F}{\partial \omega} \right|_{F=0} \sqrt{1 - u_g^2/c^2} = \frac{c}{\gamma_{u_g}} \left. \frac{\partial F}{\partial \omega} \right|_{F=0}. \quad (30)$$

Dividing the 4-vector (29) by the Lorentz scalar (30) and multiplying by c we obtain the group-velocity 4-vector,

$$u_{g,\mu} = \gamma_{u_g} (c, \mathbf{u}_g). \quad (31)$$

This 4-vector is formally identical to that of the 4-velocity (5) of a particle, so the Lorentz transformation of the group velocity \mathbf{u}_g has the form of eq. (9),

$$\mathbf{u}'_g = \frac{\mathbf{u}_g - \mathbf{v} - \mathbf{u}_{g\perp}(\mathbf{u}_g \cdot \mathbf{v})/c^2}{1 - \mathbf{u}_g \cdot \mathbf{v}/c^2}. \quad (32)$$

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⁸Fizeau’s result disagreed with the view that the æther was completely “dragged” by a moving body, as advocated by Stokes [24] and others.

⁹A recent discussion of the many repetitions of Fizeau’s experiment is [25]. Discussions of complexities in Fizeau’s experiment are given in [26, 27].

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