An Ill-Posed Problem in Rigid-Body Dynamics

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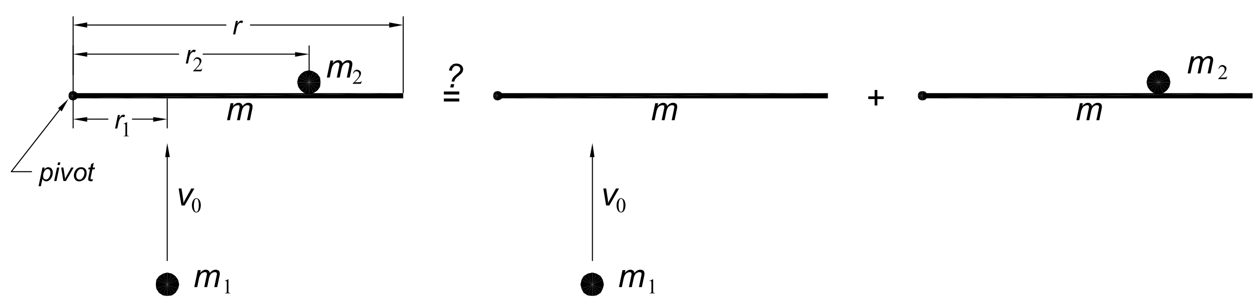
1 Problem

The concept of a rigid body is a familiar approximation in classical mechanics. That the speed of sound in infinite in a rigid body illustrates that this concept is ultimately nonphysical, but this is of little consequence in examples where all relevant speeds are small compared to the speed of light.

It is well known that apparently simple rigid-body problems have no solution. The classic example of this is a ladder leaning against a wall with friction. There are four scalar unknowns, the horizontal and vertical components of the contact forces between the ladder and the wall and the floor. But, there are only three static rigid-body equations that apply here; the total horizontal and vertical forces on the ladder, and the total torque about an axis perpendicular to the vertical plane of the ladder, are zero.

Consider the example illustrated in the figure below, in which mass m_1 has initial velocity v_0 and impact parameter r_1 with respect to the pivot point of a rod of length r and mass m. A mass m_2 is in contact with the rod at distance r_2 from the pivot. The collisions of mass m_1 with the rod and of the rod with mass m_2 are totally elastic. Can the final speeds v_1 and v_2 of the masses, and the final angular velocity ω of the rod be determined?



There are three unknowns, v_1, v_2 and ω, but only two scalar equations that govern the system: conservation of energy and conservation of angular momentum about the pivot point. Momentum is not conserved because an impulsive force must be exerted on the pivot point during the collision. Hence, in general there is no solution to this ill-posed problem.

However, we can suppose that the collisions occur in two steps: first mass m_1 collides with the rod, and then the rod collides with mass m_2. In each of these collisions there are only two unknown quantities in the final state, such that each collision can be analyzed completely. But, the hypothesis that the problem can be decomposed into two collisions is only consistent if the final velocity v_1 of mass m_1 is less than ωr_1. Otherwise the rod strikes mass m_1 after the second collision.

For what ranges of parameters does this generally ill-posed problem have a solution via the assumption that the collision occurs in two steps?

2 Solution

2.1 Analysis of the First Collision

The moment of inertial of the rod about the pivot is $I = mr^2/3$, and its angular velocity after the first collision is denoted as ω_1. The initial angular momentum of the system about the pivot is,

$$L_0 = m_1 v_0 r_1, \tag{1}$$

and the angular momentum just after the first collision is,

$$L_1 = m_1 v_1 r_1 + I\omega_1 = m_1 v_1 r_1 + \frac{mr^2\omega_1}{3}. \tag{2}$$

Conservation of angular momentum in the first collision implies that,

$$\omega_1 = \frac{3m_1 r_1}{mr^2}(v_0 - v_1). \tag{3}$$

The initial kinetic energy of the system is,

$$T_0 = \frac{m_1 v_0^2}{2}, \tag{4}$$

and the kinetic energy after the first collision is,

$$T_1 = \frac{m_1 v_1^2}{2} + \frac{I\omega_1^2}{2} = \frac{m_1 v_1^2}{2} + \frac{mr^2\omega_1^2}{6} = \frac{m_1 v_1^2}{2} + \frac{3m_1^2 r_1^2 (v_0 - v_1)^2}{2mr^2}. \tag{5}$$

Conservation of (kinetic) energy in the first collision leads to the quadratic equation,

$$(3m_1 r_1^2 + mr^2) m_1 v_1^2 - 6m_1^2 r_1^2 v_0 v_1 + (3m_1 r_1^2 - mr^2) m_1 v_0^2 = 0, \tag{6}$$

whose solution is,

$$v_1 = \frac{3m_1 r_1^2 \pm mr^2}{3m_1 r_1^2 + mr^2} v_0. \tag{7}$$

The positive sign corresponds to the trivial solution that $v_1 = v_0$, so the nontrivial solution is,

$$v_1 = \frac{3m_1 r_1^2 - mr^2}{3m_1 r_1^2 + mr^2} v_0, \qquad \omega_1 = \frac{6m_1 r_1}{3m_1 r_1^2 + mr^2} v_0. \tag{8}$$

Note that $v_1 = 0$ if the moments of inertia of mass 1 and the rod are the same.

2.2 Analysis of the Second Collision

The angular momentum of the system just before the second collision is,

$$L_1 = I\omega_1 = \frac{2mm_1 r^2 r_1}{3m_1 r_1^2 + mr^2} v_0, \tag{9}$$

and the angular momentum just after the second collision is,

$$L_2 = I\omega + m_2 v_2 r_2 = \frac{mr^2\omega}{3} + m_2 v_2 r_2. \tag{10}$$

Conservation of angular momentum in the second collision implies that,

$$\omega = \frac{6m_1 r_1}{3m_1 r_1^2 + mr^2} v_0 - \frac{3m_2 r_2}{mr^2} v_2 \,. \tag{11}$$

The kinetic energy of the system just before the second collision is,

$$T_1 = \frac{I\omega_1^2}{2} = \frac{6mm_1^2 r^2 r_1^2 v_0^2}{(3m_1 r_1^2 + mr^2)^2} \,, \tag{12}$$

and the kinetic energy after the second collision is,

$$T_2 = \frac{I\omega^2}{2} + \frac{m_2 v_2^2}{2} = \frac{mr^2\omega^2}{6} + \frac{m_2 v_2^2}{2} = \frac{6mm_1^2 r^2 r_1^2 v_0^2}{(3m_1 r_1^2 + mr^2)^2} - \frac{6m_1 m_2 r_1 r_2 v_0 v_2}{3m_1 r_1^2 + mr^2} + \frac{3m_2^2 r_2^2 v_2^2}{2mr^2} + \frac{m_2 v_2^2}{2} \,. \tag{13}$$

Conservation of (kinetic) energy in the second collision leads to the quadratic equation,

$$\frac{m_2(m_2 r^2 + 3m_2 r_2^2)}{2mr^2} v_2^2 = \frac{6m_1 m_2 r_1 r_2}{3m_1 r_1^2 + mr^2} v_0 v_2, \tag{14}$$

whose nontrivial solution is,

$$v_2 = \frac{12mm_1 r^2 r_1 r_2}{(3m_1 r_1^2 + mr^2)(3m_2 r_2^2 + mr^2)} v_0 \,, \qquad \omega = \frac{6m_1 r_1 (mr^2 - 3m_2 r_2^2)}{(3m_1 r_1^2 + mr^2)(3m_2 r_2^2 + mr^2)} v_0 \,. \tag{15}$$

Note that $\omega = 0$ if the moments of inertia of mass 2 and the rod are the same. However, this is not a complete solution unless $v_1 < 0$, which requires that $mr^2 \geq 3m_1 r_1^2$.

2.3 Consistency of the First and Second Collisions

As noted in sec. 1, the analyses in secs. 2.1-2 are complete only if the rod does not strike mass 1 again after the second collision, *i.e.*, only if $v_1 < \omega r_1$, which requires that,

$$m^2 r^4 + 3mr^2(m_1 r_1^2 + m_2 r_2^2) \geq 27 m_1 m_2 r_1^2 r_2^2. \tag{16}$$

This complicated condition is not satisfied in general, and a solution to the present problem in the context of rigid-body dynamics exists only for a subset of parameters m, m_1, m_2, r, r_1 and r_2.

The general sense of the constraint (16) is that the mass of the rod cannot be too small compared to the masses of the "point" objects 1 and 2. For if the rod has low mass it will bounce back and forth between masses 1 and 2 a large number (perhaps infinite) of times until it no longer strikes those masses.

Note that eq. (16) is an equality if the two masses and the rod have the same moment of inertia.

2.4 Solution When Kinetic Energy is Not Conserved

When kinetic energy is not conserved a useful approximation is to suppose that the ratio of the relative velocities (of the points of contact) after and before a 2-body collision has a definite value, called the coefficient of restitution.

For the first collision, we suppose that,

$$c_1 = \frac{\omega_1 r_1 - v_1}{v_0}\,, \tag{17}$$

and for the second collision,

$$c_2 = \frac{v_2 - \omega r_2}{\omega_1 r_2}\,, \tag{18}$$

are the known coefficients of restitution (with values between 0 and 1).

Then, eq. (17) can be written as,

$$\omega_1 = \frac{c_1 v_0 + v_1}{r_1}\,, \tag{19}$$

and combining this with the angular-momentum relation (3) we find,

$$v_1 = \frac{3m_1 r_1^2 - c_1 m r^2}{3m_1 r_1^2 + m r^2} v_0\,, \qquad \omega_1 = (1 + c_1)\frac{3m_1 r_1}{3m_1 r_1^2 + m r^2} v_0\,. \tag{20}$$

If $c_1 = 1$ then the results of the first collision are the same as eq. (8) when kinetic energy is conserved. If $c_1 = 0$ then mass 1 and the rod stick together, and $\omega_1 = v_1/r_1$.

Rewriting eq. (18) as,

$$\omega = \frac{v_2}{r_2} - c_2 \omega_1, \tag{21}$$

and combining this with the angular-momentum relation (11) we find,

$$v_2 = \frac{3mm_1 r^2 r_1 r_2 [2 + c_2(1 + c_1)]}{(3m_1 r_1^2 + m r^2)(3m_2 r_2^2 + m r^2)} v_0\,, \qquad \omega = \frac{6m_1 r_1 [m r^2 - 3c_2(1 + c_1) m_2 r_2^2/2]}{(3m_1 r_1^2 + m r^2)(3m_2 r_2^2 + m r^2)} v_0\,, \tag{22}$$

which reduces to eq. (15) when $c_1 = 1 = c_2$. If $c_2 = 0$ then mass 2 and the rod stick together, and $\omega = v_2/r_2$. However, the above analysis of the second collision is not correct when $c_1 = 0$ as it fails to consider mass 1 and the rod as a combined object. If $c_1 = 0 = c_2$ then $\omega = 3mr^2 v_0/(3m_1 r_1^2 + 3m_2 r_2^2 + m r^2)$, $v_1 = \omega r_1$ and $v_2 = \omega r_2$.

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