A Mechanical Model
That Exhibits a Gravitational Critical Radius

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1 Problem

A popular model at science museums (and also a science toy [1]) that illustrates how curvature can be associated with gravity consists of a surface of revolution \( r = -k/z \) with \( z < 0 \) about a vertical axis \( z \). The curvature of the surface, combined with the vertical force of Earth’s gravity, leads to an inward horizontal acceleration of \( kg/r^2 \) for a particle that slides freely on the surface in a circular, horizontal orbit.

Consider the motion of a particle that slides freely on an arbitrary surface of revolution, \( r = r(z) \geq 0 \), defined by a continuous and differentiable function on some interval of \( z \). The surface may have a nonzero minimum radius \( R \) at which the slope \( dr/dz \) is infinite. Discuss the character of oscillations of the particle about circular orbits to deduce a condition that there be a critical radius \( r_{\text{crit}} > R \), below which the orbits are unstable. That is, the motion of a particle with \( r < r_{\text{crit}} \) rapidly leads to excursions to the minimum radius \( R \), after which the particle falls off the surface.

Give one or more examples of analytic functions \( r(z) \) that exhibit a critical radius as defined above. These examples provide a mechanical analogy as to how departures of gravitational curvature from that associated with a \( 1/r^2 \) force can lead to a characteristic radius inside which all motion tends toward a singularity.

2 Solution

We work in a cylindrical coordinate system \((r, \theta, z)\) with the \( z \) axis vertical. It suffices to consider a particle of unit mass.

In the absence of friction, there is no torque on a particle about the \( z \) axis, so the angular momentum component \( J = r^2 \dot{\theta} \) about that axis is a constant of the motion, where \( \dot{} \) indicates differentiation with respect to time.

For motion on a surface of revolution \( r = r(z) \), we have \( \dot{r} = r' \dot{z} \), where \( ' \) indicates differentiation with respect to \( z \). Hence, the kinetic energy can be written as,

\[
T = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) = \frac{1}{2}[\dot{z}^2(1 + r'^2) + r^2 \dot{\theta}^2].
\]

(1)

The potential energy is \( V = gz \). Using Lagrange’s method, the equation of motion associated with the \( z \) coordinate is,

\[
\ddot{z}(1 + r'^2) + \dot{z}^2 r' r'' = -g + \frac{J^2 r'}{r^3}.
\]

(2)

For a circular orbit at radius \( r_0 \), we have,

\[
r_0^3 = \frac{J^2 r'_0}{g}.
\]

(3)
We write $\dot{\theta}_0 = \Omega$, so that $J = r_0^2 \Omega$.

For a perturbation about this orbit of the form,

$$z = z_0 + \epsilon \sin \omega t,$$  \hfill (4)

we have, to order $\epsilon$,

$$r(z) \approx r(z_0) + r'(z_0)(z - z_0)$$  \hfill (5)

$$r' \approx r'_0 + \epsilon r''_0 \sin \omega t,$$  \hfill (6)

$$\frac{1}{r^3} \approx \frac{1}{r_0^3} \left( 1 - 3 \epsilon \sin \omega t \frac{r'_0}{r_0} \right).$$  \hfill (7)

Inserting (4-7) into (2) and keeping terms only to order $\epsilon$, we obtain,

$$-\epsilon \omega^2 \sin \omega t (1 + r'_0^2) \approx -g + \frac{J^2}{r_0^3} \left( r'_0 - 3 \epsilon \sin \omega t \frac{r'_0}{r_0} + \epsilon \sin \omega t r''_0 \right).$$  \hfill (8)

From the zeroth-order terms we recover (3), and from the order-$\epsilon$ terms we find that,

$$\omega^2 = \Omega^2 \frac{3 r'_0^2 - r_0 r''_0}{1 + r'_0^2}. $$ \hfill (9)

The orbit is unstable when $\omega^2 < 0$, i.e., when,

$$r_0 r''_0 > 3 r'_0^2.$$ \hfill (10)

This condition has the interesting geometrical interpretation (noted by a referee) that the orbit is unstable wherever,

$$(1/r^2)'' < 0,$$ \hfill (11)

i.e., where the function $1/r^2$ is concave inwards.

For example, if $r = -k/z$, then $1/r^2 = z^2/k^2$ is concave outwards, $\omega^2 = J^2/(k^2 + r_0^2)$, and there is no regime of instability.

We give three examples of surfaces of revolution that satisfy condition (11).

First, the hyperboloid of revolution defined by,

$$r^2 - z^2 = R^2,$$ \hfill (12)

where $R$ is a constant. Here, $r'_0 = z_0/r_0$, $r''_0 = R^2/r_0^3$, and,

$$\omega^2 = \Omega^2 \frac{3 z_0^2 - R^2}{2 z_0^2 + R^2} = \Omega^2 \frac{3 r_0^2 - 4 R^2}{2 r_0^3 - R^2}. $$ \hfill (13)

The orbits are unstable for,

$$z_0 < \sqrt{3} R,$$ \hfill (14)

or equivalently, for,

$$r_0 < \frac{2 \sqrt{3}}{3} R = 1.1547 R \equiv r_{\text{crit}}.$$ \hfill (15)
As \( r_0 \) approaches \( R \), the instability growth time approaches an orbital period.

Another example is the Gaussian surface of revolution,

\[ r^2 = R^2 e^{z^2}, \quad (16) \]

which has a minimum radius \( R \), and a critical radius \( r_{\text{crit}} = R \sqrt[e]{e} = 1.28R \).

Our final example is the surface,

\[ r = -\frac{k}{z \sqrt[2]{1 - z^2}}, \quad (-1 < z < 0), \quad (17) \]

which has a minimum radius of \( R = 2k \), approaches the surface \( r = -k/z \) at large \( r \) (small \( z \)), and has a critical radius of \( r_{\text{crit}} = 6k/\sqrt{5} = 1.34R \).

These examples arise in a 2 + 1 geometry with curved space but flat time. As such, they are not fully analogous to black holes in 3 + 1 geometry with both curved space and curved time. Still, they provide a glimpse as to how a particle in curved spacetime can undergo considerably more complex motion than in flat spacetime.

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References