Center of Mass of a Relativistic Rolling Hoop

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1 Problem

Deduce the location of the center of mass/energy of a hoop of rest mass m_0 and rest radius a when it rolls without slipping on a horizontal surface with speed v that is not negligible compared to the speed of light c.

2 Solution

This problem is taken from sec. VII-5 of [1]. See also sec. 10,8 of [2], and [3].

2.1 Shape of the Rolling Hoop

In the first approximation the center of mass is at $x = vt$, $y = 0$, supposing the hoop rolls with speed v in the x-direction on the surface $y = -a$.

However, the top of the rolling hoop has higher speed ($\approx 2v$) than the bottom (which is instantaneously at rest in the lab frame, so the relativistic mass ($=$ energy$/c^2$, where c is the speed of light in vacuum) of an element of the hoop is larger at larger y, and the center of mass/energy is above its geometric center $(x, y) = (vt, 0)$.

We suppose that the hoop is a "rigid" body in the following sense. The $\star$ frame is defined to be the inertial frame that moves in the x-direction at velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ with respect to the lab frame, with the geometric center of the hoop at the origin in this frame. We take the hoop to be a circle of radius a in the $\star$ frame. The hoop rotates with angular velocity $\boldsymbol{\omega}^\star = -\omega\,\hat{\mathbf{z}}$ about the z-axis, where $\omega = v/a$. The relativistic mass ($=$ energy$/c^2$) $m^\star$ of the hoop in the $\star$ frame is,

$$m^\star = \gamma m_0, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,, \tag{1}$$

and is uniformly distributed in arc length $ds^\star$ around the hoop,

$$\frac{dm^\star}{ds^\star} = \frac{m^\star}{2\pi a} = \frac{\gamma m_0}{2\pi a}\,. \tag{2}$$

A point on the spinning hoop can be parameterized as,

$$x^\star = a\sin(\omega t^\star + \phi^\star), \qquad y^\star = a\cos(\omega t^\star + \phi^\star). \tag{3}$$

The lab-frame coordinates of this point are,[1]

$$x = \gamma(x^\star + vt^\star), \qquad y = y^\star, \qquad t = \gamma(t^\star + vx^\star/c^2), \tag{4}$$

[1] In the lab frame, a point on the hoop follows a trajectory of a cycloid stretched in the x-direction by factor γ, with spatial period $2\pi\gamma a$ rather than $2\pi a$.

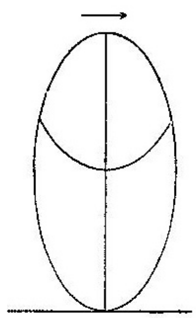
with the inverse relations,

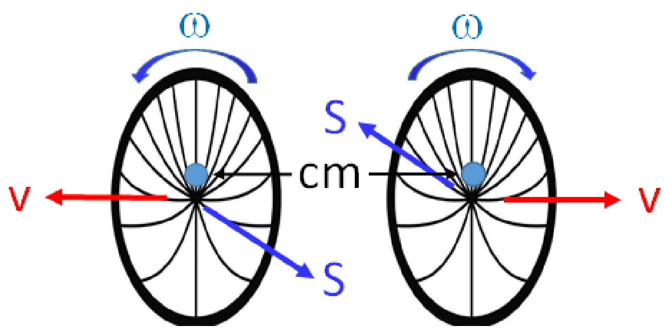
$$x^\star = \gamma(x - vt), \qquad y^\star = y, \qquad t^\star = \gamma(t - vx/c^2). \tag{5}$$

Combining eqs. (3) and (5), we see that the hoop has the form of an ellipse (Lorentz-contracted in the x-direction) in the lab frame,

$$\frac{(x - vt)^2}{(a/\gamma)^2} + \frac{y^2}{a^2} = 1. \tag{6}$$

If the hoop had two rods along orthogonal diameters in its rest frame, these rods would appear in the lab frame as shown on the left below (from [1]; the figure to the right is from [3]) at the moment when one of the rods is vertical.





The velocity of point (3) in the $\star$ frame is,

$$\mathbf{u}^\star = \boldsymbol{\omega}^\star \times \mathbf{a}^\star = \omega(y^\star, -x^\star) = \frac{v}{a}(y^\star, -x^\star). \tag{7}$$

In the lab frame this point has velocity,

$$\mathbf{u} = \frac{\mathbf{v} + \mathbf{u}^\star_x + \mathbf{u}^\star_y/\gamma}{1 + \mathbf{v}\cdot\mathbf{u}^\star/c^2} = v\frac{(1 + y^\star/a)\,\hat{\mathbf{x}} + x^\star\,\hat{\mathbf{y}}/\gamma a}{1 + v^2 y^\star/ac^2} = \frac{v}{a}\frac{(a + y)\,\hat{\mathbf{x}} + (x - vt)\,\hat{\mathbf{y}}}{1 + v^2 y/ac^2}. \tag{8}$$

The speed u is symmetric about the vertical axis $x = vt$. At the top and bottom of the rolling wheel in the lab frame, where $x = vt$ and $y = \pm a$, the instantaneous speed of a point on the hoop is $2v/(1 + v^2/c^2) < c$ and 0, respectively.[2]

2.2 Position of the Center of Mass (First Analysis)

A mass element $dm^\star$ centered on point (3) has energy-momentum 4-vector $(dE^\star, c\,d\mathbf{p}^\star) = (c^2\,dm^\star, c\,dm^\star\mathbf{u}^\star)$ in the $\star$ frame, and its energy in the lab frame is,

$$dE = c^2\,dm = \gamma(c^2\,dm^\star + v\,dm^\star u^\star_x) = \gamma c^2\,dm^\star\left(1 + \frac{v^2 y^\star}{ac^2}\right). \tag{9}$$

Thus, the total mass m of the hoop in the lab frame is (recalling eq. (2)) given by,[3,4]

$$m = \int dm = \int \frac{dm}{ds^\star}\,ds^\star = \gamma\frac{dm^\star}{ds^\star}\int\left(1 + \frac{v^2 y^\star}{ac^2}\right)ds^\star = \frac{\gamma^2 m_0}{2\pi a}\int\left(1 + \frac{v^2 y^\star}{ac^2}\right)ds^\star = \gamma^2 m_0. \tag{10}$$

[2] The point at the top of the hoop has boost $\gamma_{\rm top} = \gamma^2(1 + v^2/c^2)$.

[3] As $v \to c$, half of the rest mass becomes concentrated near the point of the top of the hoop, where the boost goes to $2\gamma^2$; the other half of the rest mass is spread around the hoop with little contribution to m.

[4] If the hoop were sliding rather than rolling, its relativistic mass in the lab frame would be γm_0.

Similarly, the center of mass/energy $\mathbf{x}_{\rm cm}$ in the lab frame is given by,

$$
\begin{aligned}
m\mathbf{x}_{\rm cm} &= \int (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}})\,dm = \int [(x^\star/\gamma + vt)\,\hat{\mathbf{x}} + y^\star\,\hat{\mathbf{y}}]\,\frac{dm}{ds^\star}\,ds^\star \\
&= \frac{\gamma^2 m_0}{2\pi a}\int [(x^\star/\gamma + vt)\,\hat{\mathbf{x}} + y^\star\,\hat{\mathbf{y}}]\left(1 + \frac{v^2 y^\star}{ac^2}\right) ds^\star \\
&= m\mathbf{v}t + \frac{mv\omega\,\hat{\mathbf{y}}}{2\pi ac^2}\int y^{\star 2}\,ds^\star = m\mathbf{v}t + \frac{ma^2\omega v\,\hat{\mathbf{y}}}{2c^2} = m\mathbf{v}t - \frac{\mathbf{S}\times\mathbf{v}}{2c^2}\,,
\end{aligned} \tag{11}
$$

where,

$$
\mathbf{S} = \frac{m\,\boldsymbol{\omega}}{2\pi a}\int (x^{\star 2} + y^{\star 2})\,ds^\star = \frac{m\,\boldsymbol{\omega}}{\pi a}\int y^{\star 2}\,ds^\star = ma^2\,\boldsymbol{\omega} \tag{12}
$$

is the "spin" angular momentum of the hoop about its geometric center.[5] Thus,

$$
\mathbf{x}_{\rm cm} = \mathbf{v}t - \frac{\mathbf{S}\times\mathbf{v}}{2mc^2} = \mathbf{x}_0 - \frac{\mathbf{S}\times\mathbf{v}}{2mc^2}\,, \tag{13}
$$

where $\mathbf{x}_0 = \mathbf{v}t$ is the position of the geometric center of the rolling hoop. That is, the center of mass/energy of the rolling hoop is shifted (upwards) relative to its geometric center.

The velocity of the center of mass is the same as the velocity of the geometric center of the hoop,

$$
\mathbf{v}_{\rm cm} = \mathbf{v}_0 = \mathbf{v}. \tag{14}
$$

The above analysis did not take into account that the rolling hoop has internal forces/stresses, In the lab frame these stresses are largest near the top of the hoop, where the speed is the greatest. Since there is mass/energy associated with the internal stresses, it seems likely that the position of the center of mass/energy is even higher above the centroid, $\mathbf{x}_0 = \mathbf{v}t$*, than indicated in eq. (13).*

We pursue this additional upward shift in the following two sections.

2.3 Position of the Center of Mass (Second Analysis)

This section follows [5]. See also [1, 6, 7], sec. 64 of [8][6] *and [9, 10, 11]. This topic has an extensive history in considerations of the quantum position operator. For a review, see [12].*

The mechanical behavior of a macroscopic subsystem can be described with the aid of its (symmetric) stress-energy-momentum 4-tensor $T^{\mu\nu}$. The quantity,

$$
P^\mu = (U/c, P^i) = (U/c, \mathbf{P}) = \int \frac{T^{0\mu}}{c}\,d\mathrm{Vol}. \tag{15}
$$

[5]The spin angular momentum (about the geometric center) of the hoop in the * frame is $\mathbf{S}^\star = m^\star a^2\,\boldsymbol{\omega}$, and $\mathbf{S} = \gamma\mathbf{S}^\star$ is the 3-vector part of Lorentz transform of the spin 4-tensor.

[6]In thermodynamics a closed subsystem can have exchange of energy, but not matter, with other subsystem, whereas an isolated subsystem has no exchange of mass/energy. The term closed system in [5, 8] corresponds to the term isolated system of thermodynamics.

describes the total energy and momentum of the subsystem, although P^μ is not truly a 4-vector unless the subsystem is isolated.[7]

The total mass/energy of the subsystem is,

$$U = \int T^{00}\, d\mathrm{Vol}, \tag{16}$$

and we define the effective mass of the subsystem as,

$$M = \frac{U}{c^2} = \int \frac{T^{00}}{c^2}\, d\mathrm{Vol} = \int \rho\, d\mathrm{Vol}, \tag{17}$$

where we define the effective mass density of the subsystem to be $\rho = T^{00}/c^2$. The center of mass/energy of the subsystem is at position,

$$x^\mu_{\mathrm{cm}} = \frac{1}{U}\int T^{00} x^\mu\ d\mathrm{Vol}, \qquad \mathbf{x}_{\mathrm{cm}} = \frac{1}{M}\int \frac{T^{00}}{c^2}\mathbf{x}\, d\mathrm{Vol}. \tag{18}$$

where $x^\mu = (ct, \mathbf{x})$, as characterizing the coordinates of the center of mass/energy of the subsystem.

In general, the lab-frame quantity x^μ_{cm} is not a 4-vector, and it is not the Lorentz transformation x^μ_0 of the quantity,

$$x^{\star\mu}_0 \equiv x^{\star\mu}_{\mathrm{cm}} = \frac{1}{U^\star}\int T^{\star 00} x^{\star\mu}\ d\mathrm{Vol}^\star, \tag{19}$$

where the $\star$ frame is the (instantaneous) frame of the subsystem in which its total 3-momentum is zero (but where the angular velocity is still $\boldsymbol{\omega}$),

$$0 = P^{\star i} = \int \frac{T^{\star 0i}}{c}\, d\mathrm{Vol}^\star. \tag{20}$$

We denote the lab-frame transform of the quantity $x^{\star\mu}_0$ as x^μ_0, which we will call the centroid.[8] Note that $x^0_0 = ct = x^0_{\mathrm{cm}}$. The velocity of the boost from the * frame to the lab frame is $\mathbf{v}_{\mathrm{cm}}$ of eq. (14). Hence, the velocity $\mathbf{v}_0$ of the centroid is the same as the velocity $\mathbf{v}_{\mathrm{cm}}$ of the center of mass/energy, even though the position $\mathbf{x}_0$ is not necessarily that same as $\mathbf{x}_{\mathrm{cm}}$.

As seen in sec. 2.2 the difference between $\mathbf{x}_{\mathrm{cm}}$ and $\mathbf{x}_0$ in the lab frame is related to the presence of angular momentum in the subsystem, so we introduce the quantity,

$$L^{\mu\nu} = \int \frac{x^\mu T^{0\nu} - x^\nu T^{0\mu}}{c}\, d\mathrm{Vol}, \tag{21}$$

as the (antisymmetric) angular momentum 4-tensor of the subsystem. Further, we introduce the "spin" angular momentum tensors, defined by,

$$S^{\mu\nu}_0 = L^{\mu\nu} - (x^\mu_0 P^\nu - x^\nu_0 P^\mu), \tag{22}$$

[7] In case of a nonisolated system, P^μ of eq. (15) has been called a "false" 4-vector [13].

[8] The coordinates x^μ_0 are called those of the proper center of mass in [8].

and,

$$S^{\mu\nu}_{\rm cm} = L^{\mu\nu} - (x^\mu_{\rm cm} P^\nu - x^\nu_{\rm cm} P^\mu), \tag{23}$$

which subtract away the angular momentum associated with the energy/momentum of the centroid, and of the center of mass/energy, respectively.

For an antisymmetric 4-tensor $A^{\mu\nu}$ we construct two 3-vectors $\mathbf{a}$ and $\tilde{\mathbf{a}}$ and according to,

$$\mathbf{a} = (a^{23}, a^{31}, a^{12}) \qquad \text{and} \qquad \tilde{\mathbf{a}} = (a^{10}, a^{20}, a^{30}). \tag{24}$$

Then, for either of the spin 4-tensors (22)-(23) we can write,

$$\mathbf{S} = \mathbf{L} - \mathbf{x}\times\mathbf{P}, \qquad \tilde{\mathbf{S}} = \tilde{\mathbf{L}} - Mc\,\mathbf{x} + ct\mathbf{P}, \qquad \text{and so} \qquad \mathbf{x} = \frac{1}{Mc}\left(\tilde{\mathbf{L}} - \tilde{\mathbf{S}} + ct\mathbf{P}\right), \tag{25}$$

where from eqs. (21),

$$\mathbf{L} = \int \mathbf{x}\times\mathbf{p}\,d\mathrm{Vol}, \qquad \text{and} \qquad \tilde{\mathbf{L}} = Mc\,\mathbf{x}_{\rm cm} - ct\mathbf{P}. \tag{26}$$

In particular, from eqs. (22) and (25)-(26) we obtain an expression for the lab-frame 3-position of the centroid,[9]

$$\mathbf{x}_0 = \mathbf{x}_{\rm cm} - \frac{\tilde{\mathbf{S}}_0}{Mc}\,. \tag{27}$$

This result was deduced in the lab frame, but it also holds in the * frame, where $\mathbf{x}^\star_0 = \mathbf{x}^\star_{\rm cm}$, so it must be that $\tilde{\mathbf{S}}^\star_0 = 0$. Then, since $\mathbf{P}^\star = 0$, we have that,

$$S^{\star\mu\nu}_0 P^\star_\nu = 0. \tag{28}$$

IF $S^{\mu\nu}_0$ and P^μ are a 4-tensor and a 4-vector, respectively, with respect to Lorentz transformations, then in the lab frame we have that[10]

$$0 = S^{\mu\nu}_0 P_\nu, \qquad \tilde{\mathbf{S}}_0\cdot\mathbf{P} = 0 \quad \text{for } \mu = 0, \qquad Mc\,\tilde{\mathbf{S}}_0 = -\mathbf{S}_0\times\mathbf{P}, \quad \text{for } \mu = 1,2,3, \tag{30}$$

and,

$$\mathbf{x}_0 = \mathbf{x}_{\rm cm} - \frac{\mathbf{S}_0\times\mathbf{P}}{M^2c^2} = \mathbf{x}_{\rm cm} - \frac{\mathbf{S}_0\times\mathbf{v}_0}{Mc^2}\,. \tag{31}$$

as in eq. (8), but now the term in $\mathbf{S}$ is twice as large, as anticipated qualitatively at the end of sec. 2.2 above.[11,12]

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[9] Using eq. (23) rather than (22) leads only to $\tilde{\mathbf{S}}_{\rm cm} = 0$, which also follows directly from eq. (23).

[10] In the * frame, $x^{\star\mu}_0 = x^{\star\mu}_{\rm cm}$, so we also have that the spin tensors are the same in this frame, $S^{\star\mu\nu}_0 = S^{\star\mu\nu}_{\rm cm}$, and in particular $\tilde{\mathbf{S}}^\star_0 = \tilde{\mathbf{S}}^\star_{\rm cm} = 0$. Since $\mathbf{P}^\star = 0$, we have that $S^{\star\mu\nu}_{\rm cm} P^\star_\nu = 0$. If $S^{\mu\nu}_{\rm cm}$ were a 4-tensor with respect to Lorentz transformations, then in the lab frame we would have that,

$$0 = S^{\mu\nu}_{\rm cm} P_\nu, \qquad \tilde{\mathbf{S}}_{\rm cm}\cdot\mathbf{P} = 0 \quad \text{for } \mu = 0, \qquad Mc\,\tilde{\mathbf{S}}_{\rm cm} = -\mathbf{S}_{\rm cm}\times\mathbf{P}, \quad \text{for } \mu = 1,2,3. \tag{29}$$

This contradicts the fact that $\tilde{\mathbf{S}}_{\rm cm} = 0$, so we infer that $S^{\mu\nu}_{\rm cm}$ is not a tensor under Lorentz transformations.

[11] In [7] it is shown that $\mathbf{S}_0\cdot\mathbf{P} = \mathbf{S}_{\rm cm}\cdot\mathbf{P} = \mathbf{L}\cdot\mathbf{P}$ and that $\mathbf{x}_0 = \mathbf{x}_{\rm cm} + \mathbf{S}_{\rm cm}\times\mathbf{P}/M_0^2c^2$ (with the + sign miswritten as a $-$).

[12] The result (31) appears in eq. (8) of [14], with $\mathbf{v}_0$ taken to be the velocity of the observer relative to the sphere, *i.e.*, $-\mathbf{v}_0$ of this note. This result was also discussed around eq. (7) of [15], with the claim that $\mathbf{x}_0$ rather than $\mathbf{x}_{\rm cm}$ is the “true” center of mass.

A spinning sphere can be regarded as a collection of hoops with a common axis of rotation. Hence, the result (31) applies to spinning spheres as well as to hoops.

For an application of this result to the interesting case of two spinning spheres, linked together by a "mechanical" rod or by gravity, and orbiting about one another, see [16].

2.4 The Hoop as a String

An interesting analysis of stresses in a spinning object has been given in [17], where it is supposed that the centripetal forces needed to sustain rotational motion are provided by strings that connect a set of more massive points which comprise the system.

In particular, for a rotating "dumbell", consisting of two masses joined by a string, the center of mass of the system would appear to oscillate in a frame in which the geometric center of the system moves with constant velocity, if one does not take into account the energy in the string.[13]

A hoop could be regarded as a loop of string, with no other point masses. Then, the methods of [17] could be applied to give a particular model of the energy-momentum-stress tensor of the hoop, to supplement the general considerations given in sec. 2.3 above.

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