On the Definition of "Hidden" Momentum

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(July 9, 2012; updated February 14, 2020)

In 1822 [1, 2], Ampère argued that the magnetic interaction between two electrical circuits could be written as a double integral over the current elements in the two circuits, with the vector integrand being along the line of centers of pairs of current elements. This insured that the total force on each circuit obeyed Newton's third law, *i.e.*, that mechanical momentum is conserved. Ampère hesitated to interpret the integrand as the force law between isolated current elements, perhaps because the integrand did not factorize. In 1845, Grassmann [5] gave an alternate form of the integrand which did factorize (into what is now called the Biot-Savart form), [1] such that the force on each circuit had the same value as in Ampère's calculation, but the forces on a pair of isolated current elements did not obey Newton's third law, *i.e.*, such that mechanical momentum would not be conserved in the interaction of isolated current elements. Maxwell (Art. 527 of [8]) favored Ampère over Grassmann (and the school of Neumann and Weber) in this matter, and was skeptical of the notion of free moving charges.[2] The concept of free moving charges was revived by J.J. Thomson beginning with speculations as to electromagnetic mass/momentum (1881) [9], and continuing with a concept of momentum stored in the electromagnetic field (1891) [11, 12], with the field-momentum density being the Poynting vector [10] divided by c^2, $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2$, where c is the speed of light in vacuum.[3,4]

Thomson was also the first to consider (1904) [18, 19, 20] an example in which an electromechanical system initially "at rest" leads to motion of part of the system if the magnetic field later drops to zero. This motion was attributed to a conversion of the initial electromagnetic field momentum into final mechanical momentum. However, Thomson did not notice that it was odd that a system initially "at rest" could end with nonzero total momentum.[5]

Thomson's 1904 paradox seems to have gone unnoticed until 1947 when Cullwick [27]-[68] discussed an electric charge and a current loop or toroidal magnet.[6] Then in 1949, Slepian

[1] This form was given by Ampère in 1825 [6], but so little emphasized by him that to this day few are aware of his effort on this.

[2] For example, sec. 65 of [7] presented what is now called the Lorentz force law, but applied it only to currents in closed circuits. In contrast, Maxwell (sec. 57 of [7]) regarded the vector potential $\mathbf{A}$ at the location of an electric charge q as providing a measure, $q\mathbf{A}/c$ (in Gaussian units, where c is the speed of light in vacuum), of electromagnetic momentum and an interpretation of Faraday's electrotonic state (secs. 60-61 of [3]. That Faraday associated with some kind of momentum with this state is hinted in sec. 1077 of [4].

[3] See [26] for verification that the total momentum of a pair of moving charges is constant to order $1/c^2$ once the electromagnetic field momentum is considered.

[4] Maxwell argued (Arts. 792-793 of [8]) that the transverse electric and magnetic fields of light wave exert a (radiation) pressure perpendicular to these field lines, which pressure is equal the energy density $u = (E^2 + B^2)/8\pi = E^2/4\pi$ (in Gaussian units) of the fields. He did not appear to make the connection that the transport of this energy at lightspeed implies a flux of energy equal to $cu = cE^2/4\pi = c\,|\mathbf{E}\times\mathbf{B}|\,/4\pi = S$, nor that the momentum density in the light wave is $p_{\rm EM} = u/c = S/c^2$. Experimental confirmation of the radiation pressure of light in 1901 [15, 17] is, however, now typically considered as evidence that electromagnetic waves carry momentum.

[5] For a review of Thomson's work on these topics, see [90].

[6] A variant of Thomson's example was discussed by Page (1932) [25], which contains net field angular

published a delightful version of it involving a (quasistatic) AC circuit [29], arguing that once electromagnetic field momentum is taken into consideration, total momentum is conserved and the device cannot be used for electromagnetic rocket propulsion.

Static systems ("at rest") can contain nonzero electromagnetic field momentum, although this tiny effect (of order $1/c^2$) has never been demonstrated experimentally. Such systems also contain a net flow of electromagnetic energy, according the relation $\mathbf{S} = c^2\,\mathbf{p}_{\rm EM}$, as perhaps first noted (1960) in sec. 5.4 of [32]. If the system is actually static, there must be a "return" flow of nonelectromagnetic energy in the system. This "return" flow of nonelectromagnetic energy will be associated with a nonzero, nonelectromagnetic momentum in the system, such that the total momentum of a system "at rest" is actually zero.

However, people were not ready to accept this logic in 1960. For example, two papers [33, 34] (published in 1964 and 1965) considered it plausible (as did Thomson in 1904 [19]) that a system "at rest" could contain nonzero total momentum (of order $1/c^2$), and that such an isolated system could transform itself to a final state in which one part was in linear motion and another part remained at rest.[7]

In contrast, Shockley (1967) [37] argued that the total momentum of an isolated electromechanical system "at rest" must be zero, and if either the electric or the magnetic field of the system later goes to zero, any motion of one part of the final system must be compensated by motion of another part with equal and opposite momentum. For this to hold, there must be some momentum "hidden" the initial system "at rest", and some "hidden" force must act within this system as the fields vanish.

Shockley's paper popularized the term "hidden" momentum (although it was no longer hidden once identified by Shockley and others),[8] and essentially all subsequent use of this term has been for electromechanical examples "at rest" that contain nonzero net electromagnetic field momentum [35, 36, 38, 39, 41, 42, 47, 48, 50, 51, 53, 54, 56, 57, 52, 58, 61, 62, 63, 69, 70, 73, 91, 92].[9] However, in these examples of systems "at rest", the nonzero net electromagnetic field momentum was not called "hidden", although it has never been detected, and is equal and opposite to the "hidden" nonelectromagnetic (*i.e.*, mechanical) momentum of the system.

It is useful to note that in this view of "hidden" momentum, it can only arise in (electro)mechanical systems with "moving parts" (Ampèrian currents), and is excluded in systems in which all magnetic effects are due to hypothetical (Gilbertian) magnetic charges or dipoles.[10]

---

momentum, but zero total (linear) momentum.

[7]Such behavior was characterized as that of a "bootstrap spaceship" in [40]. Earlier discussion of (impossible) electromagnetic spaceships had been given in [29]. See also [99].

[8]The physical origin of "hidden" momentum was identified for a particular example in [35] shortly prior to Shockley's paper.

[9]The possibility of "hidden" momentum in a gravitational system is considered [56].

The term "hidden" momentum has been used to characterize an interesting phenomenon in a binary system of spinning neutron stars [76], but the author considers that this usage is not justified [87], as also discussed in sec. 4.3 below.

[10]This result is implicit in [19], as noted in [90], and was explicity stated in [50].

1 Electromagnetic Field Momentum

We recall that the (macroscopic) electromagnetic momentum $\mathbf{P}_{\rm EM}$ of a system is most generally identified as the volume integral of $1/c^2$ times the Poynting energy-flux vector field [10],

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{1}$$

where we use Gaussian units, c is the speed of light in vacuum, and $\mathbf{E}$ and $\mathbf{B}$ are the (macroscopic) electric and magnetic fields of the system. That is,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{S}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{2}$$

as noted by Thomson [11, 12], by Heaviside [13, 23] and by Poincaré [14].[11]

The electromagnetic field momentum in quasistatic electromechanical systems can be written three other ways (see, for example, [65, 66] and Appendix B),

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \approx \int \frac{\rho\mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} \approx \int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\, d{\rm Vol} \approx \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}, \tag{3}$$

where ρ is the electric charge density, $\mathbf{A}^{\rm (C)}$ is the vector potential in the Coulomb gauge,[12] and $V^{\rm (C)}$ is the instantaneous Coulomb potential.[13,14] The second form is the electromagnetic part of the canonical momentum of the charges of the system anticipated by Faraday's electrotonic state [3, 4] and formalized by Maxwell (secs. 22-24 and 57 of [7], see also Appendix C), the third form appears to have been introduced by Furry [41], and the fourth form is due to Aharonov [47].

2 A Narrow Definition of "Hidden" Momentum

In the rest frame of the center of mass/energy of a static system (where the total momentum is zero), the "hidden" mechanical momentum was defined by Hnizdo (eq. (22) of [53]) as,[15]

$$\mathbf{P}_{\rm hidden} = -\mathbf{P}_{\rm EM} = -\int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \approx -\int \frac{\rho\mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} \approx -\int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\, d{\rm Vol} \approx -\int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}. \tag{4}$$

[11] Expressions for momentum density that combine the electromagnetic field momentum with the momentum of interacting matter in media with electric and/or magnetic polarization have been given by Abraham [16, 21] and by Minkowski [22]. We do not here enter into the "perpetual" debate (see, for example, [94]) as to the merits of the various expressions for momentum density, except for some comments in sec. 4.1.

[12] The second form of eq. (3) is actually gauge invariant if it is written in terms of the (gauge-invariant) rotational part, $\mathbf{A}_{\rm rot}$, of the vector potential, which equals the vector potential in the Coulomb gauge.

[13] $V^{\rm (C)}$ is the electric scalar potential in the Coulomb gauge.

[14] For discussion of alternative forms of the electromagnetic momentum (and energy and angular momentum) in systems with harmonic time dependence, see, for example, [75].

[15] The definition (4) leads to ambiguities in macroscopic electromagnetic media, where one can define the "electromagnetic momentum" in various ways, including the famous Abraham [16, 21] and Minkowski [22] forms. This ambiguity is another reason to seek a more general definition of "hidden" momentum.

In a frame where the center of mass/energy has nonzero velocity $\mathbf{v}_{\rm cm}$, "hidden" (mechanical) momentum given by eq. (80) of [53] as,

$$\mathbf{P}_{\rm hidden} = \mathbf{P}_{\rm mech} - M\mathbf{v}_{\rm cm}, \tag{5}$$

where $M = U/c^2$ and U is the total energy, electromagnetic plus mechanical, of the moving system.

The definition (5) appears to have been little used,[16] and essentially all discussions of "hidden" momentum until recently [37, 35, 36, 38, 39, 41, 42, 48, 50, 54, 56, 52, 73] considered only the rest frame of the center of mass/energy of the system, where definition (4) was used.

The use of the "relativistic mass" $M = U/c^2$ rather than the rest mass M_0 in eq. (5) is needed so that the momentum of a moving mass, $\gamma M_0\mathbf{v} = M\mathbf{v}$ is not considered to be "hidden".[17] A consequence of this convention is that if a static electromagnetic field has nonzero momentum $\mathbf{P}_{\rm EM}$, and field energy $U_{\rm EM}$, we can consider this field to have an equivalent mass $M = U_{\rm EM}/c^2$. The velocity of the center of mass/energy of a static field is zero, so if we apply the definition (5) to the static field, we are led to say that its momentum $\mathbf{P}_{\rm EM}$ is a "hidden" momentum. However, past usage of the term "hidden" momentum has been restricted to "mechanical" (sub)systems.

The definition (5) implies that the "overt" momentum certain mechanical systems would be called "hidden" (see, for example, [81, 101]), which suggests the need for a better definition.

3 A New Definition of "Hidden" Momentum

Recently, a definition of "hidden" momentum has been proposed [80] which can be applied to purely mechanical systems as well, where a subsystem with a specified volume can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),[18]

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol}, \tag{6}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x_\nu} = \partial_0 T^{\mu 0} + \partial_j T^{\mu j}, \tag{7}$$

[16] It was used in sec. 2 of [72].

[17] The "hidden" momentum (5) is not a component of a four-vector. It does not make sense to define a corresponding "hidden" energy, which would be $U_{\rm mech} - M_{\rm mech}c^2$, as this is identically zero.

[18] However, as noted in sec. 4 below, our interpretation of eq. (6) excludes that an "mechanical" subsystem can have "hidden" momentum unless its volume also supports a field.

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.[19,20,21]

An important qualification to the first form of eq. (6) is that the values of $\mathbf{p}$ and ρ at the boundary are the limit of those just inside the boundary.

Since f^0 involves derivatives of the stress tensor, these derivatives are infinite at (almost) any boundary of a subsystem, with a possible implication that "hidden" momentum resides at all such boundaries. This view renders the concept of "hidden" momentum to be a mathematical curiosity, with little physical significance. The attitude in this note is that such boundary delta functions are unphysical, and should not be included in the second form of eq. (6). Illustrations of analyses with and without such delta functions are given in [100, 101].

3.1 Rationale

The rationale for definition (6) begins with consideration of a subsystem of an isolated (but not necessarily bounded) system.[22,23] The subsystem occupies a specified volume (possibly unbounded) and has a bounding surface (possibly at infinity). The subsystem has a (symmetric) stress-energy-momentum 4-tensor $T^{\mu\nu}$ that is defined to be zero outside the subsystem volume. The tensor $T^{\mu\nu}$ inside the subsystem volume is not necessarily equal to the total stress-energy-momentum tensor there; for example the subsystem might be the (macroscopic) fields within the volume or the (macroscopic) matter there.

The total mass/energy of the subsystem is,

$$U = \int T^{00}\, d\text{Vol}, \tag{8}$$

and we define the effective mass of the subsystem as,

$$M = \frac{U}{c^2} = \int \frac{T^{00}}{c^2}\, d\text{Vol} = \int \rho\, d\text{Vol}, \tag{9}$$

[19]That "hidden" momentum is related to the stress tensor was pointed out in [47], eqs. (10)-(12), where only stationary systems were considered. In such cases, $f^0 = \partial T^{0i}/\partial x_i$.

[20]The definition (6) is a generalization to nonstationary systems of eq. (7) of [56], as further discussed in sec. 4.1.

[21]Since $f^0 = \partial_0 T^{00} = \partial u/\partial ct$ can be identified with $\mathbf{f}\cdot\mathbf{v}/c$, where $\mathbf{f}$ is the force density on matter with velocity $\mathbf{v}$, the last form of the definition (6) is a generalization of eq. (14) of [62]. Unfortunately, all attempts to apply that eq. (14) to examples involving "hidden" momentum [63, 64, 69, 70, 91, 92] suffer somewhat from lack of clarity as to which subsystem the terms in that equation apply.

[22]We distinguish an isolated system from a closed system in the thermodynamic sense. Neither matter nor energy flows across the bounding surface of an isolated system, while energy but not matter can flow across the boundary of a closed system.

[23]For an isolated, closed system with total stress-energy-momentum tensor $T^{\mu\nu}$, the 4-divergence of the latter is zero, $\partial T^{\mu\nu}/\partial x^\nu = 0$. If the system contains two subsystems A and B then $f^\mu_A = \partial T^{\mu\nu}_A/\partial x^\nu = -\partial T^{\mu\nu}_B/\partial x^\nu = -f^\mu_B$, where f^μ_B is the 4-force density exerted by subsystem A on B. If subsystems A and B occupy different volumes, perhaps having a surface in common, then within subvolume A, $f^\mu_A = 0$, and within subvolume B, $f^\mu_B = 0$, although there can formally be nonzero, equal and opposite force densities on the surface between the two systems (if they are in "contact".

where we define the effective mass density of the subsystem to be $\rho = T^{00}/c^2$. The center of mass/energy of the subsystem is at position,

$$\mathbf{x}_{\rm cm} = \frac{1}{M}\int \frac{T^{00}}{c^2}\mathbf{x}\, d{\rm Vol}. \tag{10}$$

Then,

$$\frac{dM}{dt} = \int \frac{\partial_0 T^{00}}{c}\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,(\mathbf{v}_b \cdot d\mathbf{Area}), \tag{11}$$

and,

$$\frac{d}{dt}(M\mathbf{x}_{\rm cm}) = \frac{dM}{dt}\mathbf{x}_{\rm cm} + M\frac{d\mathbf{x}_{\rm cm}}{dt} = \int \frac{\partial_0 T^{00}}{c}\mathbf{x}\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,\mathbf{x}\,(\mathbf{v}_b \cdot d\mathbf{Area}), \tag{12}$$

where $x^\mu = (ct, \mathbf{x})$, $\partial_\mu = \partial/\partial x^\mu = (\partial/\partial ct, \boldsymbol{\nabla})$, and $\mathbf{v}_b$ is the velocity (field) of the boundary. Hence,

$$\begin{aligned} M\mathbf{v}_{\rm cm} &= M\frac{d\mathbf{x}_{\rm cm}}{dt} = \frac{d}{dt}(M\mathbf{x}_{\rm cm}) - \frac{dM}{dt}\mathbf{x}_{\rm cm} \\ &= \int \frac{\partial_0 T^{00}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,(\mathbf{x}-\mathbf{x}_{\rm cm})\,(\mathbf{v}_b \cdot d\mathbf{Area}) \\ &= \int \frac{\partial_0 T^{00}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} + \oint_{\rm boundary} (\mathbf{x}-\mathbf{x}_{\rm cm})\,(\rho\mathbf{v}_b \cdot d\mathbf{Area}). \end{aligned} \tag{13}$$

While the stress-energy-momentum tensor for an isolated system has zero 4-divergence, this is not necessarily the case for the subsystem under consideration. Rather, the possibly nonzero 4-vector,

$$f^\mu = \partial_\nu T^{\nu\mu} = \partial_\nu T^{\mu\nu}, \tag{14}$$

describes the 4-force density exerted on the subsystem by the rest of the system. If the stress-energy-momentum tensor is nonzero just inside the boundary, the 4-force density can have delta-function contributions on the boundary (as $T^{\mu\nu}$ is defined to be zero outside the boundary), which must be considered carefully in the following. Of course, f^μ is zero outside the boundary of the subsystem.

Using eq. (14) we can write,

$$\partial_0 T^{00} = f^0 - \partial_j T^{0j}, \tag{15}$$

where both f^0 and $\partial_j T^{0j}$ can have delta functions on the boundary. Then, noting that the momentum density $\mathbf{p}$ has components $p_j = T^{0j}/c$, the volume integral in eq. (13) can be written as,

$$\int \frac{\partial_0 T^{00}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} = \int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} - \int \frac{\partial_j T^{0j}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol}$$

$$= \int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol} - \int \frac{\partial_j[T^{0j}(\mathbf{x}-\mathbf{x}_{\rm cm})]}{c}\,d{\rm Vol} + \int \frac{T^{0j}\,\partial_j \mathbf{x}}{c}\,d{\rm Vol}$$
$$= \int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol} - \oint_{\rm boundary} \frac{T^{0j}(\mathbf{x}-\mathbf{x}_{\rm cm})}{c}\,d{\rm Area}_j + \int \mathbf{p}\,d{\rm Vol}$$
$$= \int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol} - \oint_{\rm boundary} (\mathbf{x}-\mathbf{x}_{\rm cm})\,(\mathbf{p}\cdot d\mathbf{Area}) + \mathbf{P}. \tag{16}$$

Combining eqs. (13) and (16), we have,

$$\mathbf{P} = M\mathbf{v}_{\rm cm} + \oint_{\rm boundary} (\mathbf{x}-\mathbf{x}_{\rm cm})\,(\mathbf{p}-\rho\mathbf{v}_b)\cdot d\mathbf{Area} - \int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol}. \tag{17}$$

In view of the definition of "hidden" momentum given in sec. 2, we might consider either or both of the integrals in eq. (17) to be the "hidden" momentum of the subsystem.[24] To clarify this ambiguity, we consider a particular example.

3.2 Plane Electromagnetic Wave

As a test case for a new definition of "hidden" momentum we consider a plane electromagnetic wave in vacuum,

$$\mathbf{E} = E_0\,\cos(kz-\omega t)\,\hat{\mathbf{x}}, \qquad \mathbf{B} = E_0\,\cos(kz-\omega t)\,\hat{\mathbf{y}}, \tag{19}$$

in Gaussian units, where $\omega = kc$.

We take the attitude that the momentum of a plane electromagnetic wave is not "hidden".

3.2.1 The Volume Integral

It proves to be simpler to consider first volume integral in eq. (17),

$$-\int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol}. \tag{20}$$

The energy density u in this wave is,

$$u = \frac{E^2+B^2}{8\pi} = \frac{E_0^2\cos^2(kz-\omega t)}{4\pi}\,, \tag{21}$$

[24] If we restrict our attention to (sub)systems that formally have infinite extent, but negligible energy and momentum densities at "infinity", then the boundary integral can be neglected, and we can consider eq. (17) as an extension of eq. (5),

$$\mathbf{P}_{\rm hidden} = \mathbf{P} - M\mathbf{v}_{\rm cm} = -\int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\,d{\rm Vol} \qquad \text{(negligible boundary integral)}. \tag{18}$$

Our eq. (18) is eq. (3) of [76], whose authors omitted the boundary integral when integrating their eq. (2) by parts. Equation (18) is equivalent to eq. (8) of [92], where the momentum is called "internal" rather than "hidden".

and its density $\mathbf{p}$ of electromagnetic momentum is,

$$\mathbf{p} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{E_0^2\cos^2(kz-\omega t)}{4\pi c}\,\hat{\mathbf{z}} = \frac{u}{c^2}\,\mathbf{c} = \rho\,\mathbf{c}, \tag{22}$$

where $\rho = u/c^2$ is the (effective) mass/energy density, and $\mathbf{c} = c\,\hat{\mathbf{z}}$ is the velocity of the wave.

For a plane electromagnetic wave, we note that its stress tensor is,

$$T^{\mu\nu} = g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \left(\begin{array}{c|ccc} u & 0 & 0 & cp_z \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ cp_z & 0 & 0 & -u \end{array}\right). \tag{23}$$

where $T^{0z} = cp_z = u$, $T^{zz} = -E_0^2\cos^2(kz-\omega t)/4\pi = -u$, the metric tensors are,

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{24}$$

and the electromagnetic field tensors are,

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{\alpha\beta} = g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{25}$$

Then, $f^0 = \partial T^{0\nu}/\partial x^\nu = \partial u/\partial ct + \partial u/\partial z$ vanishes in the volume of integration according to eq. (21), while having δ-function terms on its surface, if we take the electromagnetic stress tensor of the volume to be zero outside it. We consider that the volume integral (20) is taken over only the interior of the volume of the subsystem, in which case the integral vanishes.

This confirms that the definition of "hidden" momentum as the volume integral in eq. (6) is consistent with plane electromagnetic waves in vacuum having zero "hidden" momentum.

3.2.2 The Surface Integral

If the "hidden" momentum of the plane wave is zero, then according to eq.(6) we should also have that,

$$\mathbf{P}_{\text{hidden}} = \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x}-\mathbf{x}_{\text{cm}})\,(\mathbf{p}-\rho\mathbf{v}_b)\cdot d\mathbf{Area} = 0. \tag{26}$$

We now consider the volume associated with the plane wave (19) to be a rectangular parallelepiped of length L along the z-direction, with area A perpendicular to $\hat{\mathbf{z}}$ and volume $V = AL$. We take the length L to be an integral number of wavelengths. The parallelepiped has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, so the velocity of its boundary is $\mathbf{v}_b = \mathbf{v}$, and we consider it at the instant when it extends over the interval $z_0 < z < L + z_0$ where $z_0 = vt$.

The total effective mass inside the parallelepiped is,

$$M = \langle \rho \rangle \, V = \frac{E_0^2 V}{8\pi c^2} \,, \tag{27}$$

since its length is an integer number of wavelengths. The averaging is formally over space, but can also be taken as the time average. The total momentum $\mathbf{P}$ inside the parallelepiped is,

$$\mathbf{P} = \langle \mathbf{p} \rangle \, V = M\mathbf{c}. \tag{28}$$

The center of mass/energy of the wave inside the parallelepiped has z-coordinate,

$$\begin{aligned} z_{\rm cm} &= \frac{\int_{z_0}^{z_0+L} z\rho \, dz}{\int_{z_0}^{z_0+L} \rho \, dz} = \frac{\int_{z_0}^{z_0+L} z \cos^2(kz - \omega t) \, dz}{L/2} = \frac{1}{L} \int_{z_0}^{z_0+L} z[1 + \cos 2(kz - \omega t)] \, dz \\ &= z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k} \,, \end{aligned} \tag{29}$$

where $z_0(t)$ is the z-coordinate of the "left" end of the parallelepiped, so the velocity of the center of mass/energy is,

$$\mathbf{v}_{\rm cm} = \mathbf{v} - \cos 2(kz_0 - \omega t) \, \mathbf{c} = \mathbf{v} + [1 - 2\cos^2(kz_0 - \omega t)] \, \mathbf{c}. \tag{30}$$

The velocity (30) of the center of mass/energy can exceed the speed of light, which indicates that this quantity does not have great physical significance in the present case of a closed, moving surface in the interior of an electromagnetic wave.

The boundary integral in eq. (26) is (for $kL = 2n\pi$),

$$\begin{aligned} &\oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm}) \, (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\mathbf{Area} \\ &= A \, \hat{\mathbf{z}} \left[(z_0 + L) - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k} \right) \right] \left[\frac{E_0^2}{4\pi c} \cos^2[k(z_0 + L) - \omega t] - \frac{E_0^2 v}{8\pi c^2} \right] \\ &\quad - A \, \hat{\mathbf{z}} \left[z_0 - \left(z_0 + \frac{L}{2} + \frac{\sin 2(kz_0 - \omega t)}{2k} \right) \right] \left[\frac{E_0^2}{4\pi c} \cos^2[kz_0 - \omega t] - \frac{E_0^2 v}{8\pi c^2} \right] \\ &= M[2\cos^2(kz_0 - \omega t) \, \mathbf{c} - \mathbf{v}]. \end{aligned} \tag{31}$$

Using the above expressions in eq. (26) we obtain,

$$\mathbf{P}_{\rm hidden} = M\mathbf{c} - M\{\mathbf{v} + [1 - 2\cos^2(kz_0 - \omega t)] \, \mathbf{c}\} - M[2\cos^2(kz_0 - \omega t)\mathbf{c} - \mathbf{v}] = 0. \tag{32}$$

This confirms that both forms of the new definition (6) of "hidden" momentum are consistent with the "hidden" momentum vanishing for a plane electromagnetic wave in vacuum.

4 Comments

An immediate consequence of the new definition (6) is that nonzero electromagnetic field momentum in a bounded, static system is called "hidden" momentum in the frame where the center of mass energy is at rest, in contrast to the narrow definition (5) which characterized only the equal and opposite mechanical momentum as "hidden". Also, in a more general frame, the quantities $M = U/c^2$ and $\mathbf{v}_{\rm cm}$ in the new definition (6) refer to a particular subsystem, while in the earlier definition (5) they refer to the entire system.

Another consequence of the new definition (6) is that an isolated system, treated as a whole, has zero total "hidden" momentum.[25]

If an isolated system is partitioned into two subsystems, we would expect their "hidden" momenta to be equal and opposite, such that the total "hidden" momentum is zero. However, it is not immediately obvious that the macroscopic integrals $\int \bar{f}^0(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}$ are equal and opposite for the two subsystems.

The second form of definition (6) indicates that a subsystem can contain "hidden" momentum only if its interior exerts a force density $-f^\mu$ on the rest of the system (such that the time component f^0 is nonzero); see also footnote 20. This implies that some other subsystem coexists within the volume of the subsystem of interest. Any subsystem that is spatially disjoint from all other subsystems will contain no "hidden" momentum.[26]

In contrast, if a system is partitioned into subsystems that can be characterized as "fields" and "matter" which occupy the same volume, the interaction between the fields and matter can be associated with "hidden" momentum.[27,28,29] In most of these examples, the "hidden" mechanical momentum is associated with moving charges, but in the example [60] (and its variant in the Appendix to [98], the "hidden" mechanical momentum is associated with the transfer of mass, without mechanical motion, from a battery to a resistor via the Poynting vector.

4.1 Electromagnetic and Mechanical Subsystems

The principle examples of subsystems that contain "hidden" momentum will be electromagnetic and mechanical subsystems with common volumes.

For an isolated, closed system with total stress-energy-momentum tensor $T^{\mu\nu}$, the 4-divergence of the latter is zero, $\partial T^{\mu\nu}/\partial x^\nu = 0$. If the system contains two subsystems A

[25] The older definition (4) considered "hidden" momentum to be mechanical but not electromagnetic. An isolated system can have nonzero, equal and opposite "hidden" mechanical and electromagnetic momenta.

[26] The usual partitioning of an all-mechanical system is into spatially disjoint subsystems, so these subsystems do not contain "hidden" momentum according to definition (6). Examples [81, 82, 83, 85, 87, 100, 101] illustrate this result.

[27] We might say that "hidden" momentum can only arise in a macroscopic field theory.

[28] Four classic examples of electromechanical systems that contain "hidden" momentum are a simple [35], or a toroidal [31], current loop in a transverse electric field, a system with both electric and magnetic dipole moments [51], and a moving capacitor [52]. For discussion by the author of these examples, see [46, 68, 79, 84, 95, 96, 97].

[29] The discussion in secs. 17.9-12 of [31] fell short of identification of the "hidden" mechanical momentum in the loop. More explicit identification of this momentum (prior to the historical introduction of the term "hidden" momentum) was made in [33, 34], but without mention of [31].

and B which occupy the same volume, then $f^\mu_A = \partial T^{\mu\nu}_A/\partial x^\nu = -\partial T^{\mu\nu}_B/\partial x^\nu = -f^\mu_B$, where f^μ_B is the 4-force density exerted by subsystem A on B. Hence, according to the last form of the definition (6), subsystems A and B have equal and opposite "hidden" momenta. In particular, if the entire system is partitioned into "electromagnetic" and "mechanical" subsystems, we have that,

$$\mathbf{P}_{\text{hidden,EM}} = -\mathbf{P}_{\text{hidden,mech}}. \tag{33}$$

For the electromagnetic subsystem the macroscopic electromagnetic energy-momentum-stress tensor (secs. 32-33 of [43], sec. 12.10B of [59]) is, in a linear medium,[30]

$$\mathsf{T}^{\mu\nu}_{EM} = \left(\begin{array}{c|c} u_{EM} & c\,\mathbf{p}_{EM} \\ \hline c\,\mathbf{p}_{EM} & -T^{ij}_{EM} \end{array}\right), \tag{34}$$

where u_{EM} is the electromagnetic field energy density, $\mathbf{p}_{EM}$ is the electromagnetic momentum density, and T^{ij}_{EM} is the 3-dimensional (symmetric) electromagnetic stress tensor. If the tensor (34) is independent of time (as, for example, in the rest frame of a medium with static charge and steady current distributions), then the quantity f^0 in eq. (6) is,

$$f^0 = \frac{\partial T^\nu}{\partial x_\nu} = c\boldsymbol{\nabla}\cdot\mathbf{p}_{\text{EM}}, \tag{35}$$

for the electromagnetic subsystem, and hence,[31]

$$\mathbf{P}_{\text{hidden,EM}} = -\int \frac{f^0}{c}(\mathbf{x}-\mathbf{x}_{\text{cm}})\,d\text{Vol} = -\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{p}_{\text{EM}})\,d\text{Vol} + \mathbf{x}_{\text{cm}}\int \boldsymbol{\nabla}\cdot\mathbf{p}_{\text{EM}}\,d\text{Vol}. \tag{36}$$

The last integral in eq. (36) transforms into a surface integral at infinity that is negligible for a system with bounded charge and current distributions. The term $-\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{p}_{\text{EM}})\,d\text{Vol}$ can be integrated by parts, with the resulting surface integral at infinity also being negligible, such that,

$$\mathbf{P}_{\text{hidden,EM}} = \int \mathbf{p}_{\text{EM}}\,d\text{Vol} = \mathbf{P}_{\text{EM}}. \tag{37}$$

Then, together with eq. (33) we have that,

$$\mathbf{P}_{\text{hidden,EM}} = \mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden,mech}}. \tag{38}$$

For a "static" case, the "visible" mechanical momentum is zero in the rest frame of the medium, and any mechanical momentum is "hidden". that is,

$$\mathbf{P}_{\text{hidden,EM}} = \mathbf{P}_{\text{EM}} = -\mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{mech}}, \tag{39}$$

and the total momentum of the system is zero,

$$\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{EM}} + \mathbf{P}_{\text{mech}} = 0. \tag{40}$$

Thus, the definition (6) is consistent with concept of "hidden" momentum as discussed by Shockley and others as explaining how/why the total momentum of an electromechanical system "at rest" is zero.

[30] For a nonlinear medium, Minkowski's stress tensor [22] is not symmetric, whereas Abraham's [21] is.

[31] For a stationary system, it is convenient to define the center of mass to be at the origin, $\mathbf{x}_{|rmcm} = 0$, in which case eq. (36) becomes eq. (7) of [56].

4.1.1 The Abraham-Minkowski Debate

The result (40) holds for any (valid) form of the electromagnetic field momentum density $\mathbf{p}_{\rm EM}$ and the associated stress-energy-momentum tensor $\mathsf{T}^{\mu\nu}_{EM}$, so the present considerations of "hidden" momentum cannot resolve the Abraham-Minkowski debate [94, 79]. That is, if one accepts either the Abraham or the Minkowski form of the stress-energy-momentum tensor, the definition (6) leads one to a computation of the "hidden" mechanical momentum that is consistent with eqs. (39)-(40).[32] One the other hand, one expects that mechanical momentum, "hidden" or not, is uniquely specifiable for a given system, so that two different values for the 'hidden" mechanical momentum cannot both be correct.

Instead, by consideration of two examples [95, 96] that contain "hidden" momentum in which the magnetic fields can be due to either convection currents or "permanent" magnetization, computation (without use of "hidden" momentum) of forces when the magnetic field goes to zero permits us to infer that neither the Abraham nor the Minkowski field momenta are "correct", and that even in systems with electric polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ the "field-only" forms (1)-(3) should be used.

4.1.2 Additional Remarks on Electromagnetic and Mechanical Subsystems

As noted in eq. (10) of [47], for a mechanical system that interacts with an electromagnetic field described by the tensor $\mathcal{F}^{\mu\nu}$ (in an overall system that may be time dependent), the 4-force density on the mechanical subsystem due to the electromagnetic subsystem is the Lorentz force,

$$f^{\mu}_{\rm mech}\ (=\partial_\nu T^{\mu\nu}_{\rm mech}) = \mathcal{F}^{\mu\nu}J_\nu = \left(\frac{\mathbf{J}\cdot\mathbf{E}}{c}, \varrho\mathbf{E} + \frac{\mathbf{J}}{c}\times\mathbf{B}\right), \qquad f^0_{\rm mech} = \frac{\mathbf{J}\cdot\mathbf{E}}{c}\,, \tag{41}$$

where the current density 4-vector is $J_\mu = (c\varrho, -\mathbf{J})$. Thus the "hidden" momentum of the mechanical subsystem is,[33]

$$\mathbf{P}^{\rm mech}_{\rm hidden} = -\int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,\left(\mathbf{r}-\mathbf{r}^{\rm mech}_{\rm cm}\right)\,d{\rm Vol} = -\int \mathbf{r}\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,d{\rm Vol} + \mathbf{r}^{\rm mech}_{\rm cm}\int\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,d{\rm Vol}. \tag{42}$$

The "hidden" momentum for the electromagnetic subsystem follows from,

$$f^0_{\rm EM} = \partial_\mu T^{0\mu}_{\rm EM}, \qquad \text{where} \qquad T^{0\mu}_{\rm EM} = (u_{\rm EM}, c\,\mathbf{p}_{\rm EM}) = \left(u_{\rm EM}, \frac{\mathbf{S}}{c}\right), \tag{43}$$

and $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ is the Poynting vector, assuming unit relative permittivity and permeability everywhere. Then,

$$f^0_{\rm EM} = \frac{1}{c}\frac{\partial u_{\rm EM}}{\partial t} + \frac{\boldsymbol{\nabla}\cdot\mathbf{S}}{c} = \frac{1}{c}\frac{\partial u_{\rm EM}}{\partial t} - \frac{1}{c}\frac{\partial u_{\rm EM}}{\partial t} - \frac{\mathbf{J}\cdot\mathbf{E}}{c} = -\frac{\mathbf{J}\cdot\mathbf{E}}{c}\,, \tag{44}$$

and,

$$\mathbf{P}^{\rm EM}_{\rm hidden} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,\left(\mathbf{r}-\mathbf{r}^{\rm EM}_{\rm cm}\right)\,d{\rm Vol} = \int \mathbf{r}\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,d{\rm Vol} - \mathbf{r}^{\rm EM}_{\rm cm}\int\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\,d{\rm Vol}. \tag{45}$$

[32] This conclusion appears to differ from that in [77].

[33] The form (42), taking $\mathbf{r}^{\rm mech}_{\rm cm} = 0$, is probably what is meant by the first equality in eq. (8) of [37], and also appears in eq. (6) of [54].

Thus,

$$\mathbf{P}^{\rm EM}_{\rm hidden} + \mathbf{P}^{\rm mech}_{\rm hidden} = (\mathbf{r}^{\rm mech}_{\rm cm} - \mathbf{r}^{\rm EM}_{\rm cm}) \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, d{\rm Vol}. \tag{46}$$

In general, $\mathbf{r}^{\rm mech}_{\rm cm}$ does not equal $\mathbf{r}^{\rm EM}_{\rm cm}$, so the total "hidden" momentum vanishes only if $\int \mathbf{J}\cdot\mathbf{E}\, d{\rm Vol} = 0$.

4.1.3 Stationary Systems

For a stationary system, $\mathbf{E} = -\boldsymbol{\nabla} V$ and $\boldsymbol{\nabla}\cdot\mathbf{J} = \partial^j J_j = 0$, so,

$$-\mathbf{J}\cdot\mathbf{E} = J_j\, \partial^j V = \partial^j (J_j V) - V \partial^j\, J_j = \partial^j (J_j V), \tag{47}$$

and,

$$\int \mathbf{J}\cdot\mathbf{E}\, d{\rm Vol} = -\int \boldsymbol{\nabla}\cdot(V\mathbf{J})\, d{\rm Vol} = -\oint (V\mathbf{J})\cdot d\mathbf{Area} = 0, \tag{48}$$

for any bounded charge density.

Hence, for stationary systems,

$$\mathbf{P}^{\rm EM}_{\rm hidden} + \mathbf{P}^{\rm mech}_{\rm hidden} = 0 \qquad ({\rm stationary}). \tag{49}$$

We also note that,

$$\begin{aligned} -\mathbf{J}\cdot\mathbf{E}\, r_i &= r_i J_j\, \partial^j V = \partial^j (r_i J_j V) - V \partial^j (r_i J_j) = \partial^j (r_i J_j V) - V J_j\, \partial^j r_i - V r_i\, \partial^j\, J_j \\ &= \partial^j (r_i J_j V) - V J_i, \end{aligned} \tag{50}$$

so,

$$P^{\rm EM}_{{\rm hidden},i} = -\int r_i\, \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, d{\rm Vol} = -\int \frac{V J_i}{c^2}\, d{\rm Vol} + \frac{1}{c^2}\int \partial^j (r_i J_j V)\, d{\rm Vol} \tag{51}$$

The latter volume integral becomes a surface integral at infinity, so is zero for any bounded current density. Then,

$$\mathbf{P}^{\rm EM}_{\rm hidden} = \mathbf{P}^{\rm EM} = \int \mathbf{r}\frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, d{\rm Vol} \quad ({\rm stationary}). \tag{52}$$

which provides a fourth prescription, in addition to those of eq. (3) (see also Appendix B), for computing the field momentum of a stationary system.

The electromagnetic momentum of a stationary system, which is equal and opposite to the "hidden" mechanical momentum, is of order $1/c^2$, which is very small compared to typical mechanical momenta. Only in isolated, stationary systems, for which $\mathbf{P}_{\rm total} = 0$, are these tiny momenta relatively prominent.

4.1.4 Quasistatic Systems

In nonstationary systems, the electromagnetic and "hidden" mechanical momentum can have terms of order $1/c^3$ (and higher). For quasistatic systems with all velocities small compared to c (and all accelerations small compared to c^2/R where R is the characteristic size of the system) these higher-order terms will be negligible, and it is a good approximation to consider the electromagnetic and "hidden" mechanical momentum only at order $1/c^2$.

For a nonstationary system, $\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A}/\partial ct$ where the vector potential $\mathbf{A}$ is of order $1/c$, and in general $\boldsymbol{\nabla} \cdot \mathbf{J} = \partial^j J_j \neq 0$, so,

$$-\mathbf{J} \cdot \mathbf{E} = J_j \, \partial^j V + J_j \partial A^j/\partial ct = \partial^j (J_j V) - V \partial^j \, J_j + \mathcal{O}(1/c^2). \tag{53}$$

For those quasistatic systems where all velocities are much less than c and $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ (which excludes moving "circuits" and time-varying charge densities),

$$\int \mathbf{J} \cdot \mathbf{E} \, d\mathrm{Vol} = -\int \boldsymbol{\nabla} \cdot (V \mathbf{J}) \, d\mathrm{Vol} = -\oint\!\!\!\!\!\!\diagup \, (V \mathbf{J}) \cdot d\mathbf{Area} = 0 + \mathcal{O}(1/c^2), \tag{54}$$

for any bounded charge density.

Hence, for some quasistationary systems,

$$\mathbf{P}^{\mathrm{EM}}_{\mathrm{hidden}} + \mathbf{P}^{\mathrm{mech}}_{\mathrm{hidden}} = 0 + \mathcal{O}(1/c^4) \qquad (\text{quasistatic, all } v \ll c, \boldsymbol{\nabla} \cdot \mathbf{J} = 0). \tag{55}$$

Systems for which eq. (55) holds (in the frame where the circuit elements are at rest) include circuits with batteries, resistors, and inductors, but not capacitors. See [60, 67] for discussion of two such examples.[34]

4.1.5 "Hidden" Angular Momentum in Quasistatic Systems

Equations (42) and (45) suggest that we identify the volume densities of "hidden" momentum as,

$$\mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right), \qquad \mathbf{p}^{\mathrm{EM}}_{\mathrm{hidden}} = \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{EM}}_{\mathrm{cm}} \right), \tag{56}$$

Then, the density of "hidden" mechanical angular momentum for the mechanical subsystem of is,[35]

$$\begin{aligned} \mathbf{l}^{\mathrm{mech}}_{\mathrm{hidden}} &= \mathbf{r} \times \mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} \times \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) = \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \mathbf{r} \times \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \\ &= \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) \times \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} = \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \times \mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}}. \end{aligned} \tag{57}$$

In this view, the total "hidden" mechanical angular momentum is,

$$\mathbf{L}^{\mathrm{mech}}_{\mathrm{hidden}} = \int \mathbf{l}^{\mathrm{mech}}_{\mathrm{hidden}} \, d\mathrm{Vol} = \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \times \int \mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} \, d\mathrm{Vol} = \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \times \mathbf{P}^{\mathrm{mech}}_{\mathrm{hidden}}, \tag{58}$$

[34] The quasistatic system of two slowly moving electric charges has nonzero field momentum, and this momentum plus the "overt" mechanical momentum is constant in time, to order $1/c^2$ [26]. Here, $\boldsymbol{\nabla} \cdot \mathbf{J} \neq 0$, and there is no "hidden" mechanical momentum.

[35] If we inferred from eq. (52) that the density of "hidden" angular momentum were $\mathbf{r} \times \pm(\mathbf{J} \cdot \mathbf{E}) \, \mathbf{r}/c^2 = 0$, there would be no "hidden" angular momentum in any system.

where $\mathbf{r}_{\rm cm}^{\rm mech}$ is the center of mass/energy of the mechanical subsystem.[36,37,38] Similarly, the total "hidden" electromagnetic angular momentum is,

$$\mathbf{L}_{\rm hidden}^{\rm EM} = \mathbf{r}_{\rm cm}^{\rm EM} \times \mathbf{P}_{\rm hidden}^{\rm EM}. \tag{60}$$

Although $\mathbf{P}_{\rm hidden}^{\rm EM} \approx -\mathbf{P}_{\rm hidden}^{\rm mech}$ in quasistatic systems where $\boldsymbol{\nabla}\cdot\mathbf{J}=0$, in general $\mathbf{r}_{\rm cm}^{\rm EM} \neq \mathbf{r}_{\rm cm}^{\rm mech}$ and $\mathbf{L}_{\rm hidden}^{\rm EM} \neq -\mathbf{L}_{\rm hidden}^{\rm mech}$. That is, there is no requirement that the total "hidden" angular momentum be zero.[39]

While the field momentum (= "hidden" field momentum) of a quasistatic system can be computed using any of four volume densities, it turns out that the field angular momentum can only be computed in general using two of those densities, of which $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})/4\pi c$ is often considered to be "the" density of field angular momentum,

$$\mathbf{L}^{\rm EM} = \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol} = \int \mathbf{r}\times\frac{\varrho\mathbf{A}}{c}\,d{\rm Vol} \neq \int \mathbf{r}\times\frac{V\mathbf{J}}{c^2}\,d{\rm Vol} \neq \mathbf{r}_{\rm cm}^{\rm EM}\times\mathbf{P}^{\rm EM} = \mathbf{L}_{\rm hidden}^{\rm EM}. \tag{61}$$

The equality of the first two forms in eq. (61) was first demonstrated in [33]. That these forms differ from $\int \mathbf{r}\times\frac{V\mathbf{J}}{c^2}\,d{\rm Vol}$ is shown in Appendix B of [93]. The fourth form, $\mathbf{L}_{\rm hidden}^{\rm EM}$, equals the first two forms only for the special case that the electric field is due to a single charge.

4.1.6 Other Alternative Expressions for "Hidden" Mechanical Momentum

As seen above, in stationary systems, and to a good approximation in quasistatic systems,

$$\mathbf{P}_{\rm hidden,mech} = -\mathbf{P}_{\rm hidden,EM} = -\mathbf{P}_{\rm EM}. \tag{62}$$

As discussed in eq. (3), in such systems the electromagnetic field momentum can be written in (at least) four ways,

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho\mathbf{A}}{c}\,d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol} = \int \frac{V\mathbf{J}}{c^2}\,d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\,d{\rm Vol}, \tag{63}$$

where the potentials V and $\mathbf{A}$ are in the Coulomb gauge, so this gives us (at least) four methods of computing the "hidden" mechanical momentum.[40] However, can the integrands

[36] The result (58) is similar to that of eq. (26) of [49]. As the result is zero when measured about the center of mass/energy of the mechanical subsystem, we could say that the "hidden" mechanical angular momentum is "orbital" rather than "intrinsic", where the latter is with respect to the center of mass/energy.

[37] The existence of nonzero "hidden" mechanical angular momentum when $\mathbf{r}_{\rm cm}^{\rm mech}$ is not at the origin is the key to the resolution of Mansuripur's paradox [78].

[38] For example, the "hidden" mechanical angular momentum of a charge q of mass M_q at distance $\mathbf{r}_q$ from a point, Ampèrian magnetic dipole $\mathbf{m}$ of mass M_m, all at rest, is, about the location of moment $\mathbf{m}$,

$$\mathbf{L}_{\rm hidden}^{\rm mech} = \frac{M_q}{M_q+M_m}\mathbf{r}_q\times\left(\frac{\mathbf{m}}{c}\times\frac{-q\mathbf{r}_q}{r_q^3}\right), \qquad \text{as [41]} \qquad \mathbf{P}_{\rm hidden}^{\rm mech} = -\mathbf{P}_{\rm EM} = \frac{\mathbf{m}\times\mathbf{E}_q}{c}\,. \tag{59}$$

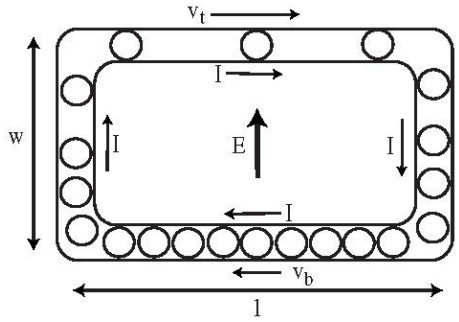
[39] For example, the "hidden" mechanical momentum of a charge q and a solenoid is essentially zero (if $m_q \ll m_{\rm solenoid}$), while the "hidden" field momentum is nonzero when the charge is not on the solenoid axis [93]. See also [45].

[40] The definition (6) gives two more methods, although the second form of eq. (6) is the same as the fourth form of eq. (63), as seen in secs. 4.1.2-3 above.

of any of the four forms in eq. (62) be regarded as the physical density of the field momentum, or the negative of the density of "hidden" mechanical momentum?

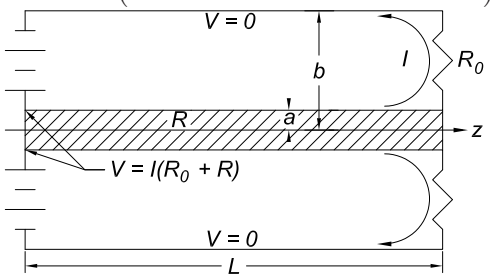
We consider that the physical density of electromagnetic field momentum is $\mathbf{E} \times \mathbf{B}/4\pi c$, so the negative of this cannot be the density of "hidden" mechanical momentum. The latter is a form of mechanical momentum, and so is associated with moving mechanical objects. In an electromechanical system, the relevant objects are the moving charges that constitute the current density $\mathbf{J}$. So, it is possible that the integrands of the third or fourth forms of eq. (63 correspond to the density of "hidden" mechanical momentum.

In a particular example [35, 73, 84] of an idealized, rectangular current loop in a uniform electric field perpendicular, as sketched below (from [73]), the "hidden" mechanical momentum is identified as the difference between the relativistic mechanical momentum of the moving charges in the top and bottom segments of the current loop. Since $\mathbf{J} \cdot \mathbf{E} = 0$ for these segments, we conclude that the density of "hidden" mechanical momentum is not $-(\mathbf{J} \cdot \mathbf{E})\mathbf{r}/c^2$.



The remaining candidate for the density of "hidden" mechanical momentum in a (quasi)stationary electromechanical system is $-V\mathbf{J}/c^2$ (for V in the Coulomb gauge). As discussed in sec. III of [73], this identification is consistent with the above example.

However, in a more subtle example [60], sketched below, the velocity of the moving charges is the same throughout the circuit, so while $V\mathbf{J}$ is nonzero except on the outer, perfect conductor, the mechanical momentum of these charges is not "hidden", and the total mechanical momentum is zero (in the frame of the circuit).



The center of mechanical mass/energy is moving to the right in this example (in the rest frame of the circuit), because the battery transfers energy to the resistor.[41] As such, the "hidden" mechanical momentum is related by the first form of eq. (6),

$$\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{mech}} - M\mathbf{v}_{\text{cm,mech}} = -M\mathbf{v}_{\text{cm,mech}}\,, \tag{64}$$

[41] That is, the system is not "at rest" in the rest frame of the circuit. If we define the lab frame to be that in which the center of mass/energy of the system is zero, then the circuit is moving to the left in the lab frame, if there are no "external" forces on the circuit.

which points to the left. In this example, there is no density of "hidden" mechanical momentum,[42] which illustrates that the concept of "hidden" momentum can be rather abstract.

4.2 "Hidden" Momentum and Relativity

"Hidden" momentum is an effect of order $1/c^2$, and as such is often described as "relativistic".[43] Yet, most discussions of "hidden" momentum are made only in the frame of an isolated system in which the center of mass/energy is at rest.[44] The definition (6) applies in any inertial frame, but it is not obvious whether the "hidden" momentum 3-vector is the spatial part of a 4-vector (or 4-tensor).

A hint of the compatibility of "hidden" momentum and relativity is given in Appendix C; the "hidden" field momentum for bounded, quasistatic systems can be written in the form,

$$\mathbf{P}^{\rm EM}_{\rm hidden} = \int \mathbf{r} \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\, d{\rm Vol} = \int \frac{\varrho \mathbf{A}}{c}\, d{\rm Vol} = \sum \frac{q\mathbf{A}}{c} \qquad \text{(quasistatic)}, \tag{65}$$

and $q\mathbf{A}$ is the spatial part of the 4-vector $qA^\mu = q(V, \mathbf{A})$.

4.3 Velocity of the Center of Mass *vs.* That of the Centroid

A pair of spinning neutron stars can contain "hidden" momentum in its gravitational field, and in the matter of the stars [76]. However, in an all-mechanical analog of this system as a linked pair of gyrostats, neither of the latter contains "hidden" momentum according to definition (6). See also [87].

This example does contain a very curious feature, that the position of the center of mass/energy of a spinning object in the lab frame is different from that of the centroid, where the latter (naïvely the geometric center) is defined as the Lorentz transformation to the lab frame of the center of mass/energy in the inertial frame where the center of mass/energy is at rest (but the object is spinning). See also [86, 88].

A possible variant of the definition (6) of "hidden" momentum is to replace $\mathbf{v}_{\rm cm}$ by $\mathbf{v}_{\rm centroid}$.

This would have the appeal of assigning a "hidden" momentum to the interesting motion in the lab frame of the neutron-star (or gyrostat-pair) system, where the combined centroid of the two spinning objects oscillates about the center of mass/energy (which latter is at rest in the lab frame) [76, 87]. However, this motion is not actually associated with any net momentum (which is zero in the lab frame).

Hence, we do not advocate use of this variant, and continue to favor the definition (6) for "hidden" momentum.

[42] The relation $\mathbf{P}_{\rm hidden} = -\int f^0(\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}/c$ is not an integral of a momentum density.

[43] See, for example, the final bullet on p. 832 of [73].

[44] An exception is [53], as noted in sec. 2.

Appendices

A Momentum of a System at "Rest"

The mechanical behavior of a macroscopic system can be described with the aid of the (symmetric) stress-energy-momentum tensor $T^{\mu\nu}$ of the system. The total energy-momentum 4-vector of the system is given by,

$$U^\mu = (U_{\rm total}, P^i_{\rm total}\, c) = \int T^{0\mu}\; d{\rm Vol}. \tag{66}$$

As first noted by Abraham [16], at the microscopic level the electromagnetic parts of $T^{\mu\nu}$ are,

$$T^{00}_{\rm EM} = \frac{E^2+B^2}{8\pi} = u_{\rm EM}\,, \tag{67}$$

$$T^{0i}_{\rm EM} = \frac{S^i}{c} = p^i_{\rm EM}\, c, \tag{68}$$

$$T^{ij}_{\rm EM} = \frac{E^iE^j + B^iB^j}{4\pi} - \delta^{ij}\frac{E^2+B^2}{8\pi}\,, \tag{69}$$

in terms of the microscopic fields $\mathbf{E}$ and $\mathbf{B}$. In particular, the density of electromagnetic momentum stored in the electromagnetic field is,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,. \tag{70}$$

The macroscopic stress tensor $T^{\mu\nu}$ also includes the "mechanical" stresses within the system, which are actually electromagnetic at the atomic level. The form (69) still holds in terms of the macroscopic fields $\mathbf{E}$ and $\mathbf{B}$ in media where $\epsilon = 1 = \mu$ such that strictive effects can be neglected. The macroscopic stresses T^{ij} are related the volume density $\mathbf{f}$ of force on the system according to,

$$f^i = \frac{\partial T^{ij}}{\partial x^j}\,. \tag{71}$$

The stress tensor $T^{\mu\nu}$ obeys the conservation law,

$$\frac{\partial T^{\mu\nu}}{\partial x_\mu} = 0, \tag{72}$$

with $x^\mu = (ct, \mathbf{x})$ and $x_\mu = (ct, -\mathbf{x})$. Once consequence of this is that the total momentum is constant for an isolated, spatially bounded system, *i.e.*,

$$\int \frac{\partial T^{\mu i}}{\partial x_\mu} = 0 = \frac{\partial}{\partial ct}\int T^{0i}\, d{\rm Vol} - \int \frac{\partial T^{ji}}{\partial x^j}\, d{\rm Vol} = \frac{dP^i_{\rm total}}{dt} - \int T^{ji}\, d{\rm Area}^j = \frac{dP^i_{\rm total}}{dt}\,. \tag{73}$$

A related result is that the total (relativistic) momentum $\mathbf{P}_{\rm total}$ of an isolated system is proportional to the velocity $\mathbf{v}_U = d\mathbf{x}_U/dt$ of the center of mass/energy of the system [39],

$$\mathbf{P}_{\rm total} = \frac{U_{\rm total}}{c^2}\mathbf{v}_U = \frac{U_{\rm total}}{c^2}\frac{d\mathbf{x}_U}{dt}\,, \tag{74}$$

where,

$$U_{\rm total} = \int T^{00}\, d{\rm Vol}, \tag{75}$$

$$P^i_{\rm total} = \frac{1}{c}\int T^{0i}\, d{\rm Vol}, \tag{76}$$

$$\mathbf{x}_U = \frac{1}{U_{\rm total}}\int T^{00}\,\mathbf{x}\, d{\rm Vol}. \tag{77}$$

That is, the total momentum of an isolated system is zero in that (inertial) frame in which the center of mass/energy is at rest.

B Electromagnetic Momentum to Order $1/c^2$

See, for example, [75] for a discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence.

Since the magnetic field $\mathbf{B}$ is always of order $1/c$ (or higher), we can calculate the electromagnetic momentum (2) to order $1/c^2$ using approximations to the electric field at zeroth order, *i.e.*, $\mathbf{E} \approx -\boldsymbol{\nabla} V^{(\rm C)}$ (the Coulomb electric field), and to the magnetic field at order $1/c$, *i.e.*, $\boldsymbol{\nabla}\times\mathbf{B} \approx 4\pi\mathbf{J}/c$ with the neglect of the displacement current. Then, as argued by Furry [41],

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \approx -\int \frac{\boldsymbol{\nabla} V^{(\rm C)}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{V^{(\rm C)}\boldsymbol{\nabla}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} - \int \frac{\boldsymbol{\nabla}\times V^{(\rm C)}\mathbf{B}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol} - \oint \frac{d\mathbf{Area}\times V^{(\rm C)}\mathbf{B}}{4\pi c} = \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol},
\end{aligned} \tag{78}$$

whenever the charges and currents are contained within a finite volume.

The following argument is due to Vladimir Hnizdo.

We can avoid use of the current density $\mathbf{J}$ and instead consider the vector potential $\mathbf{A}^{(\rm C)}$ in the Coulomb gauge, which has zero divergence,

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\rm C)}, \qquad \boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)} = 0. \tag{79}$$

In addition to well-known vector calculus relations, it is useful to define a combined operation

$$\boldsymbol{\nabla}\cdot\mathbf{ab} \equiv (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{b} + (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} = (\boldsymbol{\nabla}\cdot b_x\mathbf{a})\,\hat{\mathbf{x}} + (\boldsymbol{\nabla}\cdot b_y\mathbf{a})\,\hat{\mathbf{y}} + (\boldsymbol{\nabla}\cdot b_z\mathbf{a})\,\hat{\mathbf{z}}. \tag{80}$$

Then,

$$\begin{aligned}
\mathbf{E}\times\mathbf{B} &= \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{A}^{(\rm C)}) = \boldsymbol{\nabla}(\mathbf{A}^{(\rm C)}\cdot\mathbf{E}) - (\mathbf{A}^{(\rm C)}\cdot\boldsymbol{\nabla})\mathbf{E} - (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}^{(\rm C)} - \mathbf{A}^{(\rm C)}\times(\boldsymbol{\nabla}\times\mathbf{E}) \\
&= (\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A}^{(\rm C)} + \boldsymbol{\nabla}(\mathbf{A}^{(\rm C)}\cdot\mathbf{E}) \\
&\quad -[(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A}^{(\rm C)} + (\mathbf{A}^{(\rm C)}\cdot\boldsymbol{\nabla})\mathbf{E}] - [(\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}^{(\rm C)}] \\
&= 4\pi\rho\mathbf{A}^{(\rm C)} + \boldsymbol{\nabla}(\mathbf{A}^{(\rm C)}\cdot\mathbf{E}) - \boldsymbol{\nabla}\cdot\mathbf{E}\mathbf{A}^{(\rm C)} - \boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)}\mathbf{E},
\end{aligned} \tag{81}$$

so that,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\rho\mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} + \oint (\mathbf{A}^{\rm (C)}\cdot\mathbf{E})\; d\mathbf{Area} - \oint \mathbf{E}(\mathbf{A}^{\rm (C)}\cdot d\mathbf{Area}) - \oint \mathbf{A}^{\rm (C)}(\mathbf{E}\cdot d\mathbf{Area}) \\
&\approx \int \frac{\rho\mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol}. \qquad (82)
\end{aligned}$$

The surface integrals in eq. (82) are negligible when the charges and currents that create the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}^{\rm (C)}$ lie within a finite volume that is small compared to the volume of integration, and when radiation can be neglected. As previously noted, the electromagnetic momentum can be calculated to order $1/c^2$ with the neglect of the displacement current, which implies neglect of radiation, and justifies the approximation in eq. (82).

C Electromagnetic Potential Momentum of a Charge

Faraday anticipated [3, 4] and Maxwell noted [7] that a charge e in a vector potential $\mathbf{A}$ has a kind of electromagnetic potential momentum,[45]

$$\mathbf{P}_{\rm EM,potential} = \frac{e\mathbf{A}}{c}\,, \qquad (83)$$

in addition to its mechanical momentum, if any.[46] That is, if the vector potential drops to zero from its initial value $\mathbf{A}$, the charge experiences a "kick",

$$\Delta\mathbf{P}_{\rm mech} = \int \mathbf{F}\, dt = \int e\mathbf{E}\, dt = -\frac{e}{c}\int \frac{\partial\mathbf{A}}{\partial t}\, dt = \frac{e\mathbf{A}}{c} = \mathbf{P}_{\rm EM,potential,initial}\,. \qquad (84)$$

Since the vector potential is of order $1/c$, the potential momentum (83) is of order $1/c^2$, *i.e.*, very small for "nonrelativistic" systems.

The electromagnetic potential momentum (83) is the electromagnetic part of the canonical momentum,

$$\mathbf{P}_{\rm canonical} = \frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}} + \frac{e\mathbf{A}}{c}\,, \qquad (85)$$

of a charge e that interacts with an electromagnetic field, as described by the Lagrangian,

$$\mathcal{L} = -mc^2\sqrt{1-v^2/c^2} - e\phi + e\frac{\mathbf{v}}{c}\cdot\mathbf{A}. \qquad (86)$$

See, for example, sec. 16 of [43].

The electromagnetic potential momentum (83) is nonzero for a charge at rest, if there are charges in motion elsewhere in the system that create a nonzero vector potential at the

[45] The concept of potential momentum is reviewed in Sec. III of [74].

[46] The momentum (83) is actually gauge invariant if one writes it as $e\mathbf{A}_{\rm rot}/c$, where $\mathbf{A}_{\rm rot}$ is the rotational part of the vector potential, *i.e.*, $\boldsymbol{\nabla}\cdot\mathbf{A}_{\rm rot} = 0$. See, for example, sec. 2.3 of [75].

location of the test charge. The momentum (83), which was historically the first contribution to momentum to be identified at order $1/c^2$ (since $\mathbf{A} \propto 1/c$), could therefore be described as "hidden". However, this is not commonly done.

The term "electromagnetic potential momentum" is also not common [44, 55, 71, 89].

The electromagnetic potential momentum $e\mathbf{A}/c$ can be combined with the electrical potential energy,

$$U_{\text{EM,potential}} = eV, \tag{87}$$

where V is the electric scalar potential at the position of the charge, to form a potential-energy-momentum 4-vector,

$$U^{\mu}_{\text{EM,potential}} = (U_{\text{EM,potential}}, \mathbf{P}_{\text{EM,potential}}\, c) = e(V, \mathbf{A}) = eA^{\mu}. \tag{88}$$

where,

$$A^{\mu} = (V, \mathbf{A}), \tag{89}$$

is the electromagnetic potential 4-vector.

However, in (quasi)static cases like those considered in this note, the quantity $\int \rho A \, d\text{Vol}/c$ is equal and opposite to the "hidden" mechanical momentum of the system, and if the vector potential goes to zero, so does the "hidden" mechanical momentum, while the "overt" mechanical momentum of the system remains unchanged. There is no conversion of a "potential" momentum into an "overt" mechanical momentum, so the designation "potential momentum" is not very apt.[47]

Acknowledgment

The author thanks Daniel Cross, David Griffiths, Vladimir Hnizdo, Scott Little, Pablo Saldanha, Lev Vaidman and Daniel Vanzella for e-discussions of "hidden" momentum.

References

[1] A.M. Ampère, *Mémoire sur la détermination de la formule qui représente l'action mutuelle de deux portions infiniment petites de conducteurs voltaïques*, Ann. Chem. Phys. **20**, 398, 422 (1822), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_20_398_22.pdf

[2] Discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in R.A.R. Tricker, *Early Electrodynamics, the First Law of Circulation* (Pergamon, 1965), http://kirkmcd.princeton.edu/examples/EM/tricker_early_em.pdf
O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U. Press, 2000), http://kirkmcd.princeton.edu/examples/EM/darrigol_em_00.pdf
J.R. Hofmann, *André-Marie Ampère, Enlightenment and Electrodynamics* (Cambridge U. Press, 2006).
See also sec. IIA of J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf

[47] One might speak of $\int \rho A \, d\text{Vol}/c$ as "hidden potential momentum", but this seems awkward.

[3] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[4] M. Faraday, *Experimental Researches in Electricity.—Ninth Series*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_125_41_35.pdf

[5] H. Grassmann, *Neue Theorie der Elektrodynamik*, Ann. Phys. **64**, 1 (1845), http://kirkmcd.princeton.edu/examples/EM/grassmann_ap_64_1_45.pdf

[6] A.M. Ampère, *Théorie mathématique des Phénomènes électro-dynamiques uniquement déduite de l'Expérience*, (Paris, 1826), pp. 40-41, kirkmcd.princeton.edu/examples/EM/ampere_theorie_26.pdf
English translation in A.K.T. Assis and J.P.M.C. Chaib, *Ampère's Electrodynamics* (Aperion, 2015), p. 366, kirkmcd.princeton.edu/examples/EM/assis_ampere_15.pdf

[7] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[8] J.C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[9] J.J. Thomson, *On the Electric and Magnetic Effects produced by the Motion of Electrified Bodies*, Phil. Mag. **11**, 229 (1881), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_11_229_81.pdf

[10] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[11] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[12] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893), http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[13] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Publishing, 1893), p. 108, http://hysics.princeton.edu/~mcdonald/examples/EM/heaviside_electromagnetic_theory_1.pdf

[14] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*, http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[15] P. Lebedew, *Untersuchen über die Druckkräfte des Lichtes*, Ann. Phys. **6**, 433 (1901), http://kirkmcd.princeton.edu/examples/EM/lebedev_ap_6_433_01.pdf

[16] M. Abraham, *Prinzipien der Dynamik des Elektrons*, Ann. Phys. **10**, 105 (1903), http://physics.princeton.edu/~mcdonald/examples/EM/abraham_ap_10_105_03.pdf

[17] E.F. Nichols and G.F. Hull, *The Pressure Due to Radiation*, Phys. Rev. **17**, 26,91 (1903), http://kirkmcd.princeton.edu/examples/EM/nichols_pr_17_26_03.pdf

[18] J.J. Thomson, *Electricity and Matter* (Charles Scribner's Sons, 1904), pp. 25-35, http://kirkmcd.princeton.edu/examples/EM/thomson_electricity_matter_04.pdf

[19] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[20] J.J. Thomson, *Elements of the Mathematical Theory of Electricity and Magnetism*, 3rd ed. (Cambridge U. Press, 1904), http://kirkmcd.princeton.edu/examples/EM/thomson_EM_3rd_ed_04.pdf

[21] M. Abraham, *Zur Elelktrodynamik bewegten Körper*, Rend. Circ. Matem. Palermo **28**, 1 (1909), http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_28_1_09.pdf http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_28_1_09_english.pdf *Sull'Elettrodinamica di Minkowski*, Rend. Circ. Matem. Palermo **30**, 33 (1910), http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_30_33_10.pdf http://kirkmcd.princeton.edu/examples/EM/abraham_rcmp_30_33_10_english.pdf *Zur Frage der Symmetrie des elektromagnetischen Spannungstensors*, Ann. d. Phys. **44**, 537 (1914), http://kirkmcd.princeton.edu/examples/EM/abraham_ap_44_537_14.pdf http://kirkmcd.princeton.edu/examples/EM/abraham_ap_44_537_14_english.pdf

[22] H. Minkowski, *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper*, Nachr. Ges. Wiss. Göttingen **1**, 53 (1908), http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08.pdf http://kirkmcd.princeton.edu/examples/EM/minkowski_ngwg_53_08_english.pdf

[23] O. Heaviside, *Electromagnetic Theory*, Vol. 3 (Electrician Publishing, 1912), pp. 146-147, http://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_3.pdf

[24] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920), http://kirkmcd.princeton.edu/examples/EM/darwin_pm_39_537_20.pdf

[25] L. Page, *Momentum Considerations in Crossed Fields*, Phys. Rev. **42**, 101 (1932), http://kirkmcd.princeton.edu/examples/EM/page_pr_42_101_32.pdf

[26] L. Page and N.I. Adams, Jr., *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), http://kirkmcd.princeton.edu/examples/EM/page_ajp_13_141_45.pdf

[27] E.G. Cullwick, *Forces between Moving Charges*, Elec. Eng. **66**, 518 (1947), http://kirkmcd.princeton.edu/examples/EM/cullwick_ee_66_518_47.pdf

[28] E.G. Cullwick, *Relatively Moving Charge and Coil*, Wireless Eng. **25**, 401 (1948), http://kirkmcd.princeton.edu/examples/EM/cullwick_we_25_401_48.pdf

[29] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145, 245 (1949), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_145_49
Electrostatic Space Ship, Electrical Engineering **69**, 164, 247 (1950), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_69_164_50.pdf

[30] E.G. Cullwick, *Electromagnetic Momentum and Newton's Third Law*, Nature **170**, 425 (1952), http://kirkmcd.princeton.edu/examples/EM/cullwick_nature_170_425_52.pdf

[31] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, Green & Co., London, 1959), pp. 232-238, http://kirkmcd.princeton.edu/examples/EM/cullwick_ER_59.pdf

[32] R.M. Fano, L.J. Chu and R.B. Adler, *Electromagnetic Fields, Energy and Forces* (Wiley, 1960), http://kirkmcd.princeton.edu/examples/EM/fano_chu_adler_60.pdf

[33] G.T. Trammel, *Aharonov-Bohm Paradox*, Phys. Rev. **134**, B1183 (1964), http://kirkmcd.princeton.edu/examples/EM/trammel_pr_134_B1183_64.pdf

[34] T.T. Taylor, *Electrodynamic Paradox and the Center-of-Mass Principle*, Phys. Rev. **137**, B467 (1965), http://kirkmcd.princeton.edu/examples/EM/taylor_pr_137_B467_65.pdf

[35] P. Penfield, Jr. and H.A. Haus, *Electrodynamics of Moving Media* (MIT, 1967), p. 215, http://kirkmcd.princeton.edu/examples/EM/penfield_haus_chap7.pdf

[36] O. Costa de Beauregard, *A New Law in Electrodynamics*, Phys. Lett. **24A**, 177 (1967), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_24a_177_67.pdf
Physical Reality of Electromagnetic Potentials?, Phys. Lett. **25A**, 95 (1967), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_25a_95_67.pdf
"Hidden Momentum" in Magnets and Interaction Energy, Received 9 November 1968 Phys. Lett. **28A**, 365 (1968), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_pl_28a_365_68.pdf

[37] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents,"* Phys. Rev. Lett. **18**, 876 (1967), http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[38] H.A. Haus and P. Penfield Jr, *Forces on a Current Loop*, Phys. Lett. **26A**, 412 (1968), http://kirkmcd.princeton.edu/examples/EM/penfield_pl_26a_412_68.pdf

[39] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[40] O. Costa de Beauregard, *Magnetodynamic Effect*, Nuovo Cim. **B63**, 611 (1969), http://kirkmcd.princeton.edu/examples/EM/costa_de_beauregard_nc_b63_611_69.pdf

[41] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969), http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[42] M.G. Calkin, *Linear Momentum of the Source of a Static Electromagnetic Field*, Am. J. Phys. **39**, 513 (1971), http://kirkmcd.princeton.edu/examples/EM/calkin_ajp_39_513_71.pdf

[43] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, Oxford, 1975),
http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf
The first Russian edition appeared in 1941.
http://kirkmcd.princeton.edu/examples/EM/landau_teoria_polya_41.pdf

[44] E.J. Konopinski, *What the electromagnetic vector potential describes*, Am. J. Phys. **46**, 499 (1978), http://kirkmcd.princeton.edu/examples/EM/konopinski_ajp_46_499_78.pdf

[45] G.E. Stedman, *Observability of Static Angular Momentum*, Phys. Lett. A **81**, 15 (1981), http://kirkmcd.princeton.edu/examples/EM/stedman_pl_81a_15_81.pdf

[46] K.T. McDonald, *Stress and Momentum in a Capacitor That Moves with Constant Velocity* (Apr. 21, 1984), http://kirkmcd.princeton.edu/examples/cap_stress.pdf

[47] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052 (1988), http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf

[48] L. Vaidman, *Torque and force on a magnetic dipole*, Am. J. Phys. **58**, 978 (1990),
http://kirkmcd.princeton.edu/examples/EM/vaidman_ajp_58_978_90.pdf

[49] V. Hnizdo, *Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole*, Am. J. Phys. **60**, 242 (1992),
http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_60_242_92.pdf

[50] D.J. Griffiths, *Dipoles at rest*, Am. J. Phys. **60**, 979 (1992),
http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_60_979_92.pdf

[51] R.H. Romer, *Question #26: Electromagnetic field momentum*, Am. J. Phys. **63**, 777 (1995), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_63_777_95.pdf

[52] E. Comay, *Exposing "hidden momentum"*, Am. J. Phys. **64**, 1028 (1996),
http://kirkmcd.princeton.edu/examples/EM/comay_ajp_64_1028_96.pdf
Lorentz transformation of a system carrying Hidden Momentum, Am. J. Phys. **68**, 1007 (2000), http://kirkmcd.princeton.edu/examples/EM/comay_ajp_68_1007_00.pdf

[53] V. Hnizdo, *Hidden momentum and the electromagnetic mass of a charge and current carrying body*, Am. J. Phys. **65**, 55 (1997),
http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_65_55_97.pdf

[54] V. Hnizdo, *Hidden momentum of a relativistic fluid in an external field*, Am. J. Phys. **65**, 92 (1997), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_65_92_97.pdf

[55] M.D. Semon and J.R. Taylor, *Thoughts on the magnetic vector potential*, Am. J. Phys. **64**, 1361 (1996), http://kirkmcd.princeton.edu/examples/EM/semon_ajp_64_1361_96.pdf

[56] V. Hnizdo, *Hidden mechanical momentum and the field momentum in stationary electromagnetic and gravitational systems*, Am. J. Phys. **65**, 515 (1997), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_65_515_97.pdf

[57] V. Hnizdo, *Covariance of the total energy-momentum four-vector of a charge and current carrying macroscopic body*, Am. J. Phys. **66**, 414 (1998), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_66_414_98.pdf

[58] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

[59] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[60] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002), http://kirkmcd.princeton.edu/examples/hidden.pdf

[61] V. Hnizdo, *On linear momentum in quasistatic electromagnetic systems* (July 6, 2004), http://arxiv.org/abs/physics/0407027

[62] T.H. Boyer, *Illustrations of the relativistic conservation law for the center of energy*, Am. J. Phys. **73**, 953 (2005), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_73_953_05.pdf

[63] T.H. Boyer, *Relativity, energy flow, and hidden momentum*, Am. J. Phys. **73**, 1184 (2005), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_73_1184_05.pdf

[64] T.H. Boyer, *Connecting linear momentum and energy for electromagnetic systems*, Am. J. Phys. **74**, 742 (2006), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_74_742_06.pdf

[65] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (April 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[66] J.D. Jackson, *Relation between Interaction terms in Electromagnetic Momentum* $\int d^3\mathbf{x}\,\mathbf{E}\times\mathbf{B}/4\pi c$ *and Maxwell's* $e\mathbf{A}(\mathbf{x},t)/c$*, and Interaction terms of the Field Lagrangian* $\mathcal{L}_{\rm em}=\int d^3\mathbf{x}\,[E^2-B^2]/8\pi$ *and the Particle Interaction Lagrangian,* $\mathcal{L}_{\rm int}=e\phi-e\mathbf{v}\cdot\mathbf{A}/c$ (May 8, 2006), http://kirkmcd.princeton.edu/examples/EM/jackson_050806.pdf

[67] K.T. McDonald, *Momentum in a DC Circuit* (May 26, 2006), http://kirkmcd.princeton.edu/examples/loop.pdf

[68] K.T. McDonald, *Cullwick's Paradox: Charged Particle on the Axis of a Toroidal Magnet* (June 4, 2006), http://kirkmcd.princeton.edu/examples/cullwick.pdf

[69] T.H. Boyer, *Interaction of a point charge and a magnet: Comments on hidden mechanical momentum due to hidden nonelectromagnetic forces*, http://arxiv.org/abs/0708.3367

[70] T.H. Boyer, *Concerning "hidden momentum"*, Am. J. Phys. **76**, 190 (2008), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_76_190_08.pdf

[71] D.J. Raymond, *Potential Momentum, Gauge Theory, and Electromagnetism in Introductory Physics*, (Feb. 2, 2008), http://arxiv.org/PS_cache/physics/pdf/9803/9803023v1.pdf

[72] K. Szymański, *On the momentum of mechanical plane waves*, Physica B **403**, 2296 (2008), http://kirkmcd.princeton.edu/examples/GR/szymanski_physica_b403_2296_08.pdf

[73] D. Babson *et al.*, *Hidden momentum, field momentum, and electromagnetic impulse*, Am. J. Phys. **77**, 826 (2009), http://kirkmcd.princeton.edu/examples/EM/babson_ajp_77_826_09.pdf

[74] D.J. Griffiths, *Resource Letter EM-1: Electromagnetic Momentum*, Am. J. Phys. **80**, 7 (2012), http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_80_7_12.pdf

[75] K.T. McDonald, *Orbital and Spin Angular Momentum of Electromagnetic Fields* (Mar. 12, 2009), http://kirkmcd.princeton.edu/examples/spin.pdf

[76] S.E. Gralla, A.I. Harte, and R.M. Wald, *Bobbing and kicks in electromagnetism and gravity*, Phys. Rev. D **81**, 104012 (2010), http://kirkmcd.princeton.edu/examples/GR/gralla_prd_81_104012_10.pdf

[77] D.J. Griffiths, *Comment on "Electromagnetic momentum conservation in media"*, Ann. Phys. (N.Y.) **327**, 1290 (2012), http://kirkmcd.princeton.edu/examples/EM/griffiths_ap_327_1290_12.pdf

[78] K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012), http://kirkmcd.princeton.edu/examples/mansuripur.pdf

[79] Sec. 2.3 of K.T. McDonald, *Abraham, Minkowski and "Hidden" Momentum* (June 6, 2012), http://kirkmcd.princeton.edu/examples/abraham.pdf

[80] D. Vanzella, Private communication (June 29, 2012).

[81] K.T. McDonald, *"Hidden" Momentum in an Oscillating Tube of Water?* (June 24, 2012), http://kirkmcd.princeton.edu/examples/utube.pdf

[82] K.T. McDonald, *"Hidden" Momentum in a Link of a Moving Chain?* (June 28, 2012), http://kirkmcd.princeton.edu/examples/link.pdf

[83] K.T. McDonald, *"Hidden" Momentum in a Set of Circulating Rockets?* (June 29, 2012), http://kirkmcd.princeton.edu/examples/rockets.pdf

[84] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012), http://kirkmcd.princeton.edu/examples/penfield.pdf

[85] K.T. McDonald, *"Hidden" Momentum in a River?* (July 5, 2012), http://kirkmcd.princeton.edu/examples/river.pdf

[86] K.T. McDonald, *"Hidden" Momentum in a Spinning Sphere?* (Aug. 16, 2012), http://kirkmcd.princeton.edu/examples/spinningsphere.pdf

[87] K.T. McDonald, *"Hidden" Momentum in a Linked Pair of Gyrostats?* (Aug. 17, 2012), http://kirkmcd.princeton.edu/examples/gyrostat.pdf

[88] K.T. McDonald, *Center of Mass of a Relativistic Rolling Hoop* (Sept. 7, 2012), http://kirkmcd.princeton.edu/examples/hoop.pdf

[89] S. Barbieri, M. Cavinato and M. Giliberti, *An educational path for the magnetic vector potential and its physical implications*, Eur. J. Phys. **34**, 1209 (2013), http://kirkmcd.princeton.edu/examples/EM/barbieri_ejp_34_1209_13.pdf

[90] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014), http://kirkmcd.princeton.edu/examples/thomson.pdf

[91] T.H. Boyer, *Classical interaction of a magnet and a point charge: The Shockley-James paradox*, Phys. Rev. E **91**, 013201 (2015), http://kirkmcd.princeton.edu/examples/EM/boyer_pre_91_013201_15.pdf

[92] T.H. Boyer, *Interaction of a magnet and a point charge: Unrecognized internal electromagnetic momentum*, Am. J. Phys. **83**, 433 (2015), http://kirkmcd.princeton.edu/examples/EM/boyer_ajp_83_433_15.pdf

[93] K.T. McDonald, *Electromagnetic Field Angular Momentum of a Charge at Rest in a Uniform Magnetic Field* (Dec. 21, 2014), http://kirkmcd.princeton.edu/examples/lfield.pdf

[94] K.T. McDonald, *Bibliography on the Abraham-Minkowski Debate* (Feb. 17, 2015), http://kirkmcd.princeton.edu/examples/ambib.pdf

[95] K.T. McDonald, *"Hidden" Momentum in a Magnetized Toroid* (Mar. 29, 2015), http://kirkmcd.princeton.edu/examples/toroid_cap.pdf

[96] K.T. McDonald, *"Hidden" Momentum in a Charge, Rotating Disk* (Apr. 6, 2015), http://kirkmcd.princeton.edu/examples/rotatingdisk.pdf

[97] K.T. McDonald, *"Hidden" Momentum in an e^+e^- Pair* (Apr. 30, 2015), http://kirkmcd.princeton.edu/examples/e+e-.pdf

[98] K.T. McDonald, *Little's Paradox* (Apr. 21, 2017), http://kirkmcd.princeton.edu/examples/little.pdf

[99] K.T. McDonald, *No Bootstrap Spaceships* (Aug. 9, 2018), http://kirkmcd.princeton.edu/examples/bootstrap.pdf

[100] K.T. McDonald, *"Hidden" Momentum in a Compressed Rod?* (Feb. 11, 2020), http://kirkmcd.princeton.edu/examples/rod.pdf

[101] K.T. McDonald, *"Hidden" Momentum in an Isolated Brick?* (Feb. 14, 2020), http://kirkmcd.princeton.edu/examples/brick.pdf