On the "Helmholtz" Decomposition

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The theme of this note is that if one wishes to make a "Helmholtz" decomposition, eqs. (1)-(2) below, for electromagnetic (or other) fields one should generally not use the form (7) below of "Helmholtz' theorem", but rather use an argument of Stokes (and Maxwell) to arrive at a "Helmholtz" decomposition of the form (3)-(4). That is, eq. (7) is not generally expeditious as a method of computing a vector field, although it remains that "Helmholtz' theorem" was historically important in clarifying that a vector field can, in principle, be determined from knowledge of its curl and divergence.

On pp. 9-10 of [1] (1849), Stokes demonstrated what is now often called the "Helmholtz" decomposition, that "any" (differentiable) vector field, say $\mathbf{F}$, can be decomposed as,

$$\mathbf{F} = \mathbf{F}_{\rm irr} + \mathbf{F}_{\rm rot}, \tag{1}$$

where the irrotational and rotational components [2], $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$, obey,

$$\boldsymbol{\nabla} \times \mathbf{F}_{\rm irr} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{F}_{\rm rot} = 0. \tag{2}$$

Stokes noted that if $\boldsymbol{\nabla} \cdot \mathbf{F} = \rho$ and the scalar field V obeys Poisson's equation, $\nabla^2 V = -\rho$, then,

$$\mathbf{F}_{\rm irr} = -\boldsymbol{\nabla} V, \qquad \mathbf{F}_{\rm rot} = \mathbf{F} - \mathbf{F}_{\rm irr} = \mathbf{F} + \boldsymbol{\nabla} V, \tag{3}$$

with

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\, |\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \tag{4}$$

There is no requirement that the field $\mathbf{F}$ vanish at infinity, but the scalar potential V must be computable. Examples of fields that are nonzero at infinity with a decomposition (1)-(2) include transverse plane waves, and the fields of an infinite solenoid that carries a linearly rising current [3].

These simple results seem not to be well known. Rather, the decomposition (1)-(2) is commonly associated with Helmholtz (1858) [4], who discussed the velocity field $\mathbf{u}$ for incompressible fluids, where $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and $\mathbf{u}_{\rm irr} = 0$. Helmholtz did not cite Stokes' argument (while appearing to use a version of it), discussed our eq. (7) but not (6), and did not explicitly state the decomposition (1).

Stokes also stated, in his eq. (13) and the un-numbered equation just before his eq. (16), that $\mathbf{F}_{\rm rot} = \boldsymbol{\nabla} \times \mathbf{A}$ where $\mathbf{A}(\mathbf{r}) = \int \boldsymbol{\nabla}' \times \mathbf{F}(\mathbf{r}')\, d^3\mathbf{r}'/4\pi\, |\mathbf{r} - \mathbf{r}'|$, with the implication that,

$$\mathbf{F} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times \mathbf{A}. \tag{5}$$

It was not explicitly stated by Stokes, but it follows that (see eqs. (2) and (5a) of [4]),

$$\mathbf{F}_{\rm irr}(\mathbf{r}, t) = -\boldsymbol{\nabla} V = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{F}(\mathbf{r}', t)}{4\pi\, |\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}', \tag{6}$$

$$\mathbf{F}_{\rm rot}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{\nabla}' \times \mathbf{F}(\mathbf{r}', t)}{4\pi\, |\mathbf{r} - \mathbf{r}'|}\, d^3\mathbf{r}'. \tag{7}$$

Our eqs. (1)-(2) and (6)-(7) are commonly called "Helmholtz' theorem", although Helmholtz seems not to have stated it explicitly.

Parts of Stokes' argument were repeated by Maxwell (without attribution) in Theorem VI, p. 61 of [5] (1856). "Helmholtz theorem" was discussed, but not by name, in sec. 98, p. 33 of [6] 1881), where Gibbs wrote,

$$\mathbf{F}_{\rm irr}(\mathbf{r},t) = -\frac{1}{4\pi}\mathrm{New}\boldsymbol{\nabla}\cdot\mathbf{F} = \int \frac{(\mathbf{r}-\mathbf{r}')\ [\boldsymbol{\nabla}'\cdot\mathbf{F}(\mathbf{r}',t)]}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|^3}\, d^3\mathbf{r}', \tag{8}$$

$$\mathbf{F}_{\rm rot}(\mathbf{r},t) = \frac{1}{4\pi}\mathrm{Lap}\boldsymbol{\nabla}\times\mathbf{F} = -\int \frac{(\mathbf{r}-\mathbf{r}')\times[\boldsymbol{\nabla}'\times\mathbf{F}(\mathbf{r}',t)]}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|^3}\, d^3\mathbf{r}'. \tag{9}$$

Heaviside gave versions of both our eqs. eqs. (3)-(4) and our eqs. (6)-(7), using his operator "Pot", in [7] (1892), reprinted as §134, p. 206 of [8]. Another early mention of "Helmholtz' theorem", also not by name, is in §23, pp. 188-191 of [9] (1896), where Stokes' paper was cited. An early reference to "Helmholtz theorem" by that name is in § 71, p. 155 of [10] (1909).

Relatively short "proofs" of "Helmholtz' theorem" typically assume that the field $\mathbf{F}$ vanishes at infinity. Some people claim that eqs. (6)-(7) hold only for fields that vanish faster than $1/r$ at infinity (see, for example, Appendix B of [11]) [12], although it was shown in 1905 [16] that "Helmholtz' theorem" holds for any (differentiable) vector field that vanishes at infinity (*i.e.*, falls off as $1/r^\epsilon$ with $\epsilon > 0$); for a review in English, see [17, 18]. This does not exclude that the theorem holds for many fields that are nonzero at infinity.

As an application of the above (not made by Maxwell), we consider the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, which obey the Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{10}$$

where ρ is the electric charge density and ϵ_0 is the permittivity of the vacuum. Then, we have that,

$$\mathbf{E}_{\rm irr} = \mathbf{E}_{\rm Coulomb}, \qquad \mathbf{E}_{\rm rot} = \mathbf{E} - \mathbf{E}_{\rm Coulomb}, \qquad \mathbf{B}_{\rm irr} = 0, \qquad \mathbf{B}_{\rm rot} = \mathbf{B}, \tag{11}$$

where the instantaneous Coulomb field is related by,

$$\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm Coulomb} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla}\times\mathbf{E}_{\rm Coulomb} = 0, \tag{12}$$

$$\mathbf{E}_{\rm Coulomb} = -\boldsymbol{\nabla}V_{\rm Coulomb}, \qquad V_{\rm Coulomb}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{4\pi\epsilon_0}\, d^3\mathbf{r}'. \tag{13}$$

For static fields, $\mathbf{E} = \mathbf{E}_{\rm Coulomb} = \mathbf{E}_{\rm irr}$, and $\mathbf{E}_{\rm rot} = 0$. For source-free electromagnetic fields, such as infinite plane waves, $\mathbf{E}_{\rm irr} = 0$ and $\mathbf{E}_{\rm rot} = \mathbf{E}$. The "Helmholtz" decomposition for the electric field of an oscillating, "point" (Hertzian) electric dipole is given in Appendix A.1 of [19].

References

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[2] The irrotational component of a 3-vector is sometimes labeled "longitudinal", "parallel" or "polar", and the rotational component is sometimes labeled "circuital", "solenoidal" or "transverse".

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