The Helmholtz Decomposition and the Coulomb Gauge

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1 Problem

Helmholtz showed in 1858 [1] (in a hydrodynamic context) that any vector field, say $\mathbf{E}$, that vanishes suitably quickly at infinity can be decomposed as,[1,2,3]

$$\mathbf{E} = \mathbf{E}_{\rm irr} + \mathbf{E}_{\rm rot}, \tag{1}$$

where the irrotational and rotational components $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$ obey,[4]

$$\boldsymbol{\nabla} \times \mathbf{E}_{\rm irr} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{E}_{\rm rot} = 0. \tag{2}$$

For the case that $\mathbf{E}$ is the electric field, discuss the relation of the Helmholtz decomposition to use of the Coulomb gauge.[5,6]

2 Solution

The Helmholtz decomposition (1)-(2) is an artificial split of the vector field $\mathbf{E}$ into two parts, which parts have interesting mathematical properties.

We recall that in electrodynamics the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be related to a scalar potential V and a vector potential $\mathbf{A}$ according to (in SI units),

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \tag{3}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{4}$$

This results in another decomposition of the electric field $\mathbf{E}$ which might be different from that of Helmholtz. Here, we explore the relation between these two decompositions.

[1] The essence of this decomposition was anticipated by Stokes (1849) in secs. 5-6 of [2].

[2] For a review of the Helmholtz' decomposition in nonelectromagnetic contexts, see [3].

[3] Radiation fields, which fall off as $1/r$ at large distance r from their (bounded) source, do fall off sufficiently quickly for Helmholtz' decomposition to apply, as reviewed in [4]. Doubts as to this were expressed in [5], but see [6]. See Appendix A for the Helmholtz decomposition of the electromagnetic fields of a Hertzian ("point", oscillating) dipole, which illustrates that such a decomposition is readily made when radiation is present.

[4] The irrotational component is sometimes labeled "longitudinal" or "parallel", and the rotational component is sometimes labeled "solenoidal" or "transverse".

[5] Vector plane waves $\mathbf{E}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ do not vanish "suitably quickly" at infinity, so care is required in applying the Helmholtz decomposition $\mathbf{E}_{\rm irr} = (\mathbf{E}\cdot\mathbf{k})\,\hat{\mathbf{k}}$, $\mathbf{E}_{\rm rot} = \mathbf{E} - \mathbf{E}_{\rm irr}$ of this mathematically useful, but physically unrealistic class of fields. See, for example, sec. 2.4.2 of [7].

[6] In case of coupled fields like $\mathbf{E}$ and $\mathbf{B}$ of electromagnetism, their expressions in terms of retarded potentials can be cast into forms somewhat similar to the Helmholtz decomposition. See [8].

We also recall that the potentials V and $\mathbf{A}$ are not unique, but can be redefined in a systematic way such that the fields $\mathbf{E}$ and $\mathbf{B}$ are invariant under such redefinition. A particular choice of the potentials is called a choice of gauge, and the relations (3)-(4) are said to be gauge invariant.[7]

Returning to Helmholtz' decomposition, we note that he also showed how,

$$\mathbf{E}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}', \qquad \text{and} \qquad \mathbf{E}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}', \tag{5}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Time does not appear in eq. (5), which indicates that the vector field $\mathbf{E}$ at some point $\mathbf{r}$ (and some time t) can be reconstructed from knowledge of its vector derivatives, $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$, over all space (at the same time t).[8] The main historical significance of the Helmholtz decomposition (1) and (5) was in showing that Maxwell's equations, which give prescriptions for the vector derivatives $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$, are mathematically sufficient to determine the field $\mathbf{E}$. Since $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho_{\rm total}/\epsilon_0$ and $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, the fields $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$ involve instantaneous action at a distance and should not be regarded as physically real. This illustrates how gauge invariance in necessary, but not sufficient, for electromagnetic fields to correspond to "reality".[9,10,11]

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[7] The gauge transformation $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$, $V \to V - \partial\chi/\partial(ct)$, leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged. A consequence of this is that when the vector potential is decomposed as $\mathbf{A} = \mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot}$, the rotational part is unchanged by transformations where $\nabla^2\chi \neq 0$. That is, in such cases, $\mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot} \to (\mathbf{A}_{\rm irr} + \boldsymbol{\nabla}\chi) + \mathbf{A}_{\rm rot}$, where the term in parenthesis is the irrotational part of the transformed vector potential, so the rotational part, $\mathbf{A}_{\rm rot}$, is unchanged by the gauge transformation.

On the other hand, if $\nabla^2\chi = 0$ everywhere, then the gauge transformation is $\mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot} \to \mathbf{A}_{\rm irr} + (\mathbf{A}_{\rm rot} + \boldsymbol{\nabla}\chi)$, which leaves the irrotational part of $\mathbf{A}$ unchanged.

[8] If the field $\mathbf{E}$ is known only within a finite volume V, bounded by a closed surface S, then the Helmholtz decomposition (1) and (5) becomes,

$$\mathbf{E}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla}\left(\int_{\rm V} \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \int_{\rm S} \frac{\hat{\mathbf{n}} \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Area}'\right), \tag{6}$$

$$\mathbf{E}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \left(\int_{\rm V} \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \int_{\rm S} \frac{\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Area}'\right), \tag{7}$$

where $\hat{\mathbf{n}}$ is the inward unit normal vector on the surface S. That is,

$$\mathbf{E} = -\boldsymbol{\nabla}V \qquad \text{only if} \qquad \boldsymbol{\nabla} \times \mathbf{E} = 0 \quad \text{in V, and} \quad \hat{\mathbf{n}} \times \mathbf{E} = 0 \quad \text{on S}, \tag{8}$$

$$\mathbf{E} = \boldsymbol{\nabla} \times \mathbf{A} \qquad \text{only if} \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = 0 \quad \text{in V, and} \quad \hat{\mathbf{n}} \cdot \mathbf{E} = 0 \quad \text{on S}. \tag{9}$$

Neglect of the conditions on the surface S can lead to error, as remarked in [9]-[12].

[9] The forms (5) are not the only possible representations of $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$. For example, we could write,

$$\mathbf{E}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla}\left[\int \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + C\right], \qquad \text{and} \qquad \mathbf{E}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \left[\int \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \boldsymbol{\nabla}\chi\right], \tag{10}$$

for any constant C and any differentiable scalar function χ, without changing the values of $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$.

Also, since $[\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')]/R = \boldsymbol{\nabla}' \cdot [\mathbf{E}(\mathbf{r}')/R] + \boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}')/R$ and $[\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')]/R = \boldsymbol{\nabla}' \times [\mathbf{E}(\mathbf{r}')/R] + \boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}')/R$ (noting that $\boldsymbol{\nabla}(1/R) = -\mathbf{R}/R^3 = -\boldsymbol{\nabla}'(1/R)$), if $\mathbf{E}$ falls off sufficiently quickly at large R we can rewrite $\mathbf{E}_{\rm rot}$ as,

$$\mathbf{E}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}', \qquad \mathbf{E}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \frac{\mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}'. \tag{11}$$

[10] See Appendix A for an application of the Helmholtz decomposition to Hertzian dipole radiation.

[11] The Helmholtz decomposition leads to interesting interpretations of the momentum and angular momentum associated with electromagnetic fields [7].

The Helmholtz decomposition (1) and (5) can be rewritten as,

$$\mathbf{E} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times \mathbf{F}, \tag{12}$$

where,

$$V(\mathbf{r}) = \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\text{Vol}', \qquad \text{and} \qquad \mathbf{F}(\mathbf{r}) = \int \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\text{Vol}'. \tag{13}$$

It is consistent with usual nomenclature to call V a scalar potential and $\mathbf{F}$ a vector potential. That is, Helmholtz decomposition lends itself to an interpretation of fields as related to derivatives of potentials.

When the vector field $\mathbf{E}$ is the electric field, it also obeys Maxwell's equations, two of which are (in SI units and for media where the permittivity is ϵ_0),

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{14}$$

where ρ is the electric charge density and $\mathbf{B}$ is the magnetic field.

If we insert these physics relations into eq. (13), we find,

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 R}\, d\text{Vol}', \tag{15}$$

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial}{\partial t} \int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\text{Vol}'. \tag{16}$$

The scalar potential (15) is calculated from the instantaneous charge density, which is exactly the prescription (44) of the Coulomb gauge. That is, Helmholtz + Maxwell implies use of the Coulomb-gauge prescription for the scalar potential.

However, eq. (16) for the vector potential $\mathbf{F}$ does not appear to be that of the usual procedures associated with the Coulomb gauge. Comparing eqs. (12)-(13) and (16), we see that we can introduce another vector potential $\mathbf{A}$ which obeys,

$$\boldsymbol{\nabla} \times \mathbf{F} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{17}$$

such that,

$$\mathbf{A}(\mathbf{r}) = \boldsymbol{\nabla} \times \int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\text{Vol}', \tag{18}$$

and,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \tag{3}$$

which is the usual way the electric field is related to a scalar potential V and a vector potential $\mathbf{A}$. Note also that eq. (18) obeys the Coulomb gauge condition (40) that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.[12]

Thus, the Helmholtz decomposition (1) and (5) of the electric field E is equivalent to the decomposition (3) in terms of a scalar and a vector potential, provided those potentials are calculated in the Coulomb gauge.[13]

[12]See, for example, sec. 3 of [13].

[13]The fields $\mathbf{E}_{\text{irr}} = -\boldsymbol{\nabla} V - \partial \mathbf{A}_{\text{irr}}/\partial(ct)$ and $\mathbf{E}_{\text{rot}} = -\partial \mathbf{A}_{\text{rot}}/\partial(ct)$ can be deduced from scalar potential V and vector potential $\mathbf{A} = \mathbf{A}_{\text{irr}} + \mathbf{A}_{\text{rot}}$ in any gauge, but only in the Coulomb gauge is $\mathbf{A}_{\text{irr}}^{(C)} = 0$ such that the Helmholtz decomposition has the simple form $\mathbf{E}_{\text{irr}} = -\boldsymbol{\nabla} V^{(C)}$ and $\mathbf{E}_{\text{rot}} = -\partial \mathbf{A}^{(C)}/\partial t$.

Using various vector calculus identities, we have,

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \boldsymbol{\nabla}\times\int\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\,d\mathrm{Vol}' = \int\boldsymbol{\nabla}\frac{1}{R}\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi}\,d\mathrm{Vol}' = -\int\boldsymbol{\nabla}'\frac{1}{R}\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi}\,d\mathrm{Vol}' \\
&= \int\frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\,d\mathrm{Vol}' + \int\boldsymbol{\nabla}'\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\,d\mathrm{Vol}' \\
&= \int\frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\,d\mathrm{Vol}' - \oint d\mathbf{Area}'\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi R} = \int\frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\,d\mathrm{Vol}', \qquad (19)
\end{aligned}$$

provided $\mathbf{B}$ vanishes sufficiently quickly at infinity. In view of the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we recognize eq. (19) as the Helmholtz decomposition $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ for the magnetic field.[14]

We can go further by invoking the Maxwell equation,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,, \qquad (23)$$

where $\mathbf{J}$ is the current density vector, the medium is assumed to have permeability μ_0, and c is the speed of light, so that,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{R}\,d\mathrm{Vol}' + \frac{\partial}{\partial t}\int\frac{\mathbf{E}(\mathbf{r}')}{4\pi c^2 R}\,d\mathrm{Vol}'. \qquad (24)$$

This is not a useful prescription for calculation of the vector potential, because the second term of eq. (24) requires us to know $\mathbf{E}(\mathbf{r}')/c^2$ to be able to calculate $\mathbf{E}(\mathbf{r})$.[15] But, c^2 is a big number, so $\mathbf{E}/c^2$ is only a "small" correction, and perhaps can be ignored.[16] If we do so,

[14]We can verify the consistency of eqs. (18) and (19) by taking the curl of the latter. For this, we note that,

$$\boldsymbol{\nabla}\times\frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R} = -(\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}'))\times\boldsymbol{\nabla}\left(\frac{1}{4\pi R}\right) = (\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}'))\times\boldsymbol{\nabla}'\left(\frac{1}{4\pi R}\right). \qquad (20)$$

The i-component of this is,

$$\begin{aligned}
&\epsilon_{i,j,k}\epsilon_{jlm}(\partial'_l B_m)\partial'_k(1/4\pi R) = \delta^{ki}_{lm}(\partial'_l B_m)\partial'_k(1/4\pi R) = (\partial'_k B_i)\partial'_k(1/4\pi R) - (\partial'_i B_k)\partial'_k(1/4\pi R) \\
&= \partial'_k[B_i\partial'_k(1/4\pi R)] - B_i\nabla'^2(1/4\pi R) - \partial'_k[(1/4\pi R)\,\partial'_i B_k] + (1/4\pi R)\,\partial'_i\boldsymbol{\nabla}'\cdot\mathbf{B} \\
&= B_i(\mathbf{r}')\,\delta^3(\mathbf{r}-\mathbf{r}') + \partial'_k[B_i\partial'_k(1/4\pi R) - (1/4\pi R)\,\partial'_i B_k]. \qquad (21)
\end{aligned}$$

The volume integral of this gives $\mathbf{B}(\mathbf{r})$ plus a surface integral that vanishes if the magnetic field falls off sufficiently quickly at large distances. That is, $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$ for the vector potentials given by eqs. (18) and (19).

We could also proceed by taking the curl of eq. (18), noting that,

$$\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\right) - \mathbf{B}(\mathbf{r}')\,\nabla^2\left(\frac{1}{4\pi R}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\right) + \mathbf{B}(\mathbf{r}')\,\delta^3(\mathbf{r}-\mathbf{r}'). \qquad (22)$$

Then, integrating this over $d\mathrm{Vol}'$ gives $\mathbf{B}(\mathbf{r})$ plus a surface integral that vanishes for magnetic fields that fall off sufficiently quickly at large distances. So, again we find that $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$.

This footnote is due to Vladimir Hnizdo. See also [14].

[15]Using the Helmholtz decomposition for $\mathbf{E}$ in eq. (24) permits us to proceed without knowing $\mathbf{E}$, provided we know the charge density ρ and the time derivative $\partial\mathbf{B}/\partial t$, which is no improvement conceptually.

[16]See [15] for an argument that the second integral of eq. (24) vanishes in the quasistatic approximation.

then,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\,d\text{Vol}', \tag{25}$$

which is the usual instantaneous prescription for the vector potential due to steady currents. Thus, it appears that practical use of the Helmholtz decomposition + Maxwell's equations is largely limited to quasistatic situations, where eqs. (15) and (25) are sufficiently accurate.[17]

Of course, we exclude wave propagation and radiation in this approximation. We can include radiation and wave propagation if we now invoke the usual prescription, eqs. (46)-(47) of Appendix B, for the vector potential in the Coulomb gauge. However, this prescription does not follow very readily from the Helmholtz decomposition, which is an instantaneous calculation.

Note that in the case of practical interest when the time dependence of the charges and currents is purely sinusoidal at angular frequency ω, i.e., $e^{-i\omega t}$, the Lorenz gauge condition [17] (39) becomes,

$$V = -\frac{ic}{k}\boldsymbol{\nabla}\cdot\mathbf{A}. \tag{26}$$

In this case it suffices to calculate only the vector potential $\mathbf{A}$, and then deduce the scalar potential V, as well as the fields $\mathbf{E}$ and $\mathbf{B}$, from $\mathbf{A}$.

However, neither the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ nor the Lorenz gauge condition (39) suffices, in general, for a prescription in which only the scalar potential V is calculated, and then $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ are deduced from this. Recall that the Helmholtz decomposition tells us how the vector field $\mathbf{A}$ can be reconstructed from knowledge of both $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$. The gauge conditions tell us only $\boldsymbol{\nabla}\cdot\mathbf{A}$, and we lack a prescription for $\boldsymbol{\nabla}\times\mathbf{A}$ in terms of V.

[In 1 dimension, $\boldsymbol{\nabla}\times\mathbf{A} = 0$, so in 1-dimensional problems we can deduce everything from the scalar potential V plus the gauge condition.[18] But life in 3 dimensions is more complicated!]

A Appendix: Helmholtz Decomposition of Hertzian Dipole Radiation

The electric and magnetic fields of an ideal, point Hertzian electric dipole can be written (in Gaussian units) as,

$$\mathbf{E} = k^2 p(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\times\hat{\mathbf{r}}\frac{\cos(kr-\omega t)}{r} + p[3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \hat{\mathbf{p}}]\left[\frac{\cos(kr-\omega t)}{r^3} + \frac{k\sin(kr-\omega t)}{r^2}\right], \tag{27}$$

$$\mathbf{B} = \mathbf{B}_{\text{rot}} = k^2 p(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\left[\frac{\cos(kr-\omega t)}{r} - \frac{\sin(kr-\omega t)}{kr^2}\right], \tag{28}$$

[17] A version of Helmholtz' theorem in which the integrands involve retarded quantities has been given in [16], especially sec. 4.

[18] For additional discussion of electrodynamics in one spatial dimension, see [20].

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, $\mathbf{p} = p\cos\omega t\,\hat{\mathbf{p}}$ is the electric dipole moment vector, $\omega = 2\pi f$ is the angular frequency, $k = \omega/c = 2\pi/\lambda$ is the wave number and c is the speed of light [21, 22].

The irrotational part of the electric field is the instantaneous field of the electric dipole,

$$\mathbf{E}_{\rm irr} = p[3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}]\frac{\cos\omega t}{r^3}\,. \tag{29}$$

Thus, the rotational part of the electric field is,

$$\begin{aligned}\mathbf{E}_{\rm rot} = \mathbf{E} - \mathbf{E}_{\rm irr} &= k^2 p(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\times\hat{\mathbf{r}}\frac{\cos(kr-\omega t)}{r}\\ &+p[3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \hat{\mathbf{p}}]\left[\frac{\cos(kr-\omega t)-\cos\omega t}{r^3} + \frac{k\sin(kr-\omega t)}{r^2}\right].\end{aligned} \tag{30}$$

Both fields $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$ have instantaneous terms.

The flow of energy in the electromagnetic field is described by the Poynting vector $\mathbf{S}$, so the Helmholtz decomposition leads us to write,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = \frac{c}{4\pi}\mathbf{E}_{\rm irr}\times\mathbf{B}_{\rm rot} + \frac{c}{4\pi}\mathbf{E}_{\rm rot}\times\mathbf{B}_{\rm rot}. \tag{31}$$

Using eqs. (28)-(29), we have that,

$$\mathbf{S}_1 = \frac{ck^2p^2}{4\pi}[(3\cos^2\theta - 1)\,\hat{\mathbf{r}} - 2\cos\theta\,\hat{\mathbf{p}}]\cos\omega t\left[\frac{\cos(kr-\omega t)}{r^4} - \frac{\sin(kr-\omega t)}{kr^5}\right], \tag{32}$$

where θ is the angle between vectors $\mathbf{r}$ and $\mathbf{p}$. Similarly,

$$\begin{aligned}\mathbf{S}_2 &= \frac{c}{4\pi}\left\{k^4p^2\sin^2\theta\,\hat{\mathbf{r}}\left[\frac{\cos^2(kr-\omega t)}{r^2} - \frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right]\right.\\ &+k^2p^2[(3\cos^2\theta-1)\,\hat{\mathbf{r}} - 2\cos\theta\,\hat{\mathbf{p}}]\left[\frac{\cos^2(kr-\omega t)-\sin^2(kr-\omega t)}{r^4}\right.\\ &\quad+\cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3} - \frac{1}{kr^5}\right)\\ &\quad\quad\left.\left.-\cos\omega t\left(\frac{\cos(kr-\omega t)}{r^4} - \frac{\sin(kr-\omega t)}{kr^5}\right)\right]\right\}.\end{aligned} \tag{33}$$

Neither $\mathbf{S}_1$ nor $\mathbf{S}_2$ describes the flow of energy at an identifiable speed, so the Helmholtz decomposition, which is based on present source terms, does not seem well suited to a general characterization of the flow of energy in electromagnetic fields.

We can restrict our attention to the region very close to the source, where $kr \ll 1$ and we have,

$$\mathbf{S}_1(kr\ll 1) = \frac{ck^2p^2}{4\pi}[(3\cos^2\theta-1)\,\hat{\mathbf{r}} - 2\cos\theta\,\hat{\mathbf{p}}]\left(\frac{\cos^2\omega t}{r^4} + \frac{\cos\omega t\sin\omega t}{kr^5}\right), \tag{34}$$

and,

$$\mathbf{S}_2(kr \ll 1) = \frac{c}{4\pi}\left[k^4p^2\sin^2\theta\,\hat{\mathbf{r}}\left(\frac{\cos^2\omega t}{r^2}+\frac{\cos\omega t\sin\omega t}{kr^3}\right)\right.$$
$$\left.+k^2p^2[(3\cos^2\theta-1)\,\hat{\mathbf{r}}-2\cos\theta\,\hat{\mathbf{p}}]\left(\frac{k\cos\omega t\sin\omega t}{r^3}-\frac{\sin^2\omega t}{r^4}\right)\right]. \tag{35}$$

Here, the separation of the total Poynting vector $\mathbf{S}$ into $\mathbf{S}_1$ and $\mathbf{S}_2$ is cleaner than for large kr, but, to this author, this separation is still not associated with any crisp physical insight.

We can also consider only the time average of eqs. (32)-(33),

$$\langle\mathbf{S}_1\rangle = \frac{ck^2p^2}{8\pi}[(3\cos^2\theta-1)\,\hat{\mathbf{r}}-2\cos\theta\,\hat{\mathbf{p}}]\left(\frac{\cos kr}{r^4}-\frac{\sin kr}{kr^5}\right), \tag{36}$$

and,

$$\langle\mathbf{S}_2\rangle = \frac{c}{8\pi}\frac{k^4p^2\sin^2\theta}{r^2}\,\hat{\mathbf{r}}-\langle\mathbf{S}_1\rangle\,. \tag{37}$$

Again, there seems to be little physical insight associated with this decomposition.

B Appendix: Coulomb Gauge

The relations (in SI units),

$$\mathbf{E} = -\boldsymbol{\nabla}V-\frac{\partial\mathbf{A}}{\partial t}, \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{38}$$

between the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and the potentials V and $\mathbf{A}$ permits various conventions (gauges) for the potentials. According to Helmholtz, the vector potential is determined by its curl, $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$, and by its divergence, $\boldsymbol{\nabla}\cdot\mathbf{A}$. So, to complete the determination of $\mathbf{A}$, given $\mathbf{B}$, we must specify its divergence, which latter is called the gauge condition.

One popular choice is the Lorenz gauge [17],

$$\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}. \tag{39}$$

In situations with steady charge and current distributions (electrostatics and magnetostatics), $\partial V/\partial t = 0$, so the condition (39) reduces to,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0 \qquad \text{(Coulomb)}. \tag{40}$$

Even in time-dependent situations it is possible to define the vector potential to obey eq. (40), which has come to be called the Coulomb gauge condition.

Using eq. (38) together with the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho/\epsilon_0$ leads to,

$$\boldsymbol{\nabla}^2V+\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{\rho}{\epsilon_0}\,, \tag{41}$$

and the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \partial \mathbf{E}/\partial c^2 t$ leads to,

$$\boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right). \tag{42}$$

Thus, in the Coulomb gauge (40), eq. (41) becomes Poisson's equation,

$$\boldsymbol{\nabla}^2 V^{(\mathrm{C})} = -\frac{\rho}{\epsilon_0}, \tag{43}$$

which has the formal solution,

$$V^{(\mathrm{C})}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{R}\, d\mathrm{Vol}' \qquad \text{(Coulomb)}, \tag{44}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, in which changes in the charge distribution ρ instantaneously affect the potential V at any distance.[19]

For completeness, a formal solution for the vector potential in the Coulomb gauge follows from eq. (42) with $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$,

$$\boldsymbol{\nabla}^2 \mathbf{A}^{(\mathrm{C})} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{(\mathrm{C})}}{\partial t^2} = -\mu_0 \mathbf{J} + \epsilon_0\mu_0 \frac{\partial \boldsymbol{\nabla} V^{(\mathrm{C})}}{\partial t} \equiv -\mu_0 \tilde{\mathbf{J}}, \tag{45}$$

using the method of retarded potentials [17, 18, 19],[20]

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\tilde{\mathbf{J}}(\mathbf{r}', t' = t - R/c)}{R}\, d\mathrm{Vol}'. \tag{46}$$

The vector $\tilde{\mathbf{J}}$ can be re-expressed using a Helmholtz decomposition of the current density $\mathbf{J} = \mathbf{J}_{\mathrm{irr}} + \mathbf{J}_{\mathrm{rot}}$. Then, recalling eq. (5), the continuity equation $\boldsymbol{\nabla} \cdot \mathbf{J} + \partial\rho/\partial t = 0$, and eq. (44),

$$\mathbf{J}_{\mathrm{irr}}(\mathbf{r}, t) = -\boldsymbol{\nabla}\int \frac{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}', t)}{4\pi R}\, d\mathrm{Vol}' = \boldsymbol{\nabla}\int \frac{\partial \rho(\mathbf{r}', t)}{\partial t}\frac{d\mathrm{Vol}'}{4\pi R} = \epsilon_0 \frac{\partial \boldsymbol{\nabla} V^{(\mathrm{C})}(\mathbf{r}, t)}{\partial t}, \tag{47}$$

and we see from eqs. (11) and (45) that,

$$\tilde{\mathbf{J}} = \mathbf{J}_{\mathrm{rot}}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \frac{\mathbf{J}(\mathbf{r}', t)}{4\pi R}\, d\mathrm{Vol}' = \mathbf{J} - \mathbf{J}_{\mathrm{irr}} = \mathbf{J} - \epsilon_0 \frac{\partial \boldsymbol{\nabla} V^{(\mathrm{C})}(\mathbf{r}, t)}{\partial t}, \tag{48}$$

which is called the transverse current in [24]. Note that $\mathbf{J}_{\mathrm{irr}}$ and $\mathbf{J}_{\mathrm{rot}}$ are nonzero throughout all space (except perhaps on certain curves and surfaces) if $\mathbf{J}$ is nonzero anywhere.

While the Coulomb-gauge vector potential (46) would appear to propagate (in vacuum) with the speed of light, this is not so in general, as illustrated by the case of a Hertzian electric

[19] It is possible to choose gauges for the electromagnetic potentials such that some of their components appear to propagate at any specified velocity v [24, 25, 26]. One can also choose that the scalar potential be zero [27], or have no time dependence [28] such that all time dependence of the electric field is associated with that of the vector potential.

[20] For a static current density, eq. (46) reduces to eq. (25).

dipole, reviewed in Appendix B of [27]. And, the part of the electric field, $-\partial\mathbf{A}^{(C)}/\partial t$, derived from it has pieces that propagate instantaneously, as needed to cancel the instantaneous behavior of the part, $-\boldsymbol{\nabla}V^{(C)}$, derived from the Coulomb-gauge scalar potential (44). For additional discussion, see, for example, [29]-[32].

Unless the geometry of the problem is such that the rotation current density $\mathbf{J}_{\rm rot}$ is easy to calculate, use of the Coulomb gauge is technically messier than the use of the Lorenz gauge, in which case the (retarded) potentials are given by the retarded potentials,

$$V^{(L)}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t'=t-R/c)}{R}\,d\mathrm{Vol}' \qquad \textit{(Lorenz)}, \tag{49}$$

$$\mathbf{A}^{(L)}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t'=t-R/c)}{R}\,d\mathrm{Vol}' \qquad \textit{(Lorenz)}, \tag{50}$$

where $R = |\mathbf{r}-\mathbf{r}'|$.

Analyses of circuits are typically performed in the quasistatic approximation that effects of wave propagation and radiation can be neglected. In this case, the speed of light is taken to be infinite, so that the Lorenz gauge condition (39) is equivalent to the Coulomb gauge condition (40), and the potentials are calculated from the instantaneous values of the charge and current distributions. As a consequence, gauge conditions are seldom mentioned in "ordinary" circuit analysis.

B.1 Alternative Forms of the Coulomb-Gauge Potentials

While the potentials (44) and (46) can be considered to be the standard forms for the Coulomb gauge, they are not the only possible ones.

If the gauge-transformation function χ obeys Laplace's equation, $\nabla^2\chi = 0$, then the vector potential of the gauge transformation $\mathbf{A}^{(C)} \to \mathbf{A}^{(C)} + \boldsymbol{\nabla}\chi$, $V^{(C)} \to V^{(C)} - \partial\chi/\partial t$, also satisfies the Coulomb-gauge condition (40).[21]

In quasistatic examples of charge and current densities within a bounded volume, and where radiation can be ignored, the standard potentials (44) and (46) go to zero at large distances. Then, all of the alternative Coulomb-gauge potentials, generated by a gauge function χ that obeys Laplace's equation do not go to zero at infinity in all directions. This follows from the uniqueness theorem for solutions to Laplace's equation with either Dirichlet or Neumann boundary conditions (see, for example sec. 1.9 of [22]), since the trivial case $\chi = 0$ has both χ and its derivatives equal to zero (at large distances). So, when one adds the constraint that the Coulomb-gauge potentials must vanish at infinity, then the standard forms (44) and (46) are the only such solutions (if indeed they vanish at infinity).[22]

[21] For example, the gauge functions $\chi \pm Bxy/2$ lead from the axially symmetric vector potential of a uniform magnetic field $B\,\hat{\mathbf{z}}$ inside an axially symmetric current distribution to the so-called "Landau" potentials, which are nonzero at infinity. See also sec. 2.1 of [33].

[22] It is claimed in eq. (B.26), p. 17, of [34] that $\mathbf{A}_{\rm rot}$ (called $\mathbf{A}_\perp$ there) is gauge invariant, since the Fourier transform of the gauge transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$ is $\mathbf{A}'_{\mathbf{k}} = \mathbf{A}_{\mathbf{k}} + i\mathbf{k}\,\chi_{\mathbf{k}}$. This makes it appear that the term $i\mathbf{k}\,\chi_{\mathbf{k}}$ contributes only to the irrotational part of $\mathbf{A}'$, since the Fourier transform of $\mathbf{A}_{\rm irr}$ is $\mathbf{A}_{\mathbf{k},{\rm irr}} = (\hat{\mathbf{k}}\cdot\mathbf{A}_{\mathbf{k}})\,\hat{\mathbf{k}}$ (as in eq. (B.14a) of [34]), such that $\mathbf{A}'_{\rm rot} = \mathbf{A}_{\rm rot}$. However, $\hat{\mathbf{k}}$ is undefined for $\mathbf{k} = 0$, such that the Fourier component $\mathbf{A}_{\mathbf{k}=\mathbf{0}}$ is entirely rotational. Then, if $\nabla^2\chi = 0$, its Fourier transform is $0 = -k^2\chi_{\mathbf{k}} = i\mathbf{k}\cdot(i\mathbf{k}\,\chi_{\mathbf{k}})$, such that $i\mathbf{k}\,\chi_{\mathbf{k}}$ can be nonzero for $\mathbf{k} = \mathbf{0}$, in which case $\boldsymbol{\nabla}\chi$ contributes to $\mathbf{A}'_{\rm rot}$ and this differs from $\mathbf{A}_{\rm rot}$.

B.2 Coulomb-Gauge Potentials for Hertzian Dipole Radiation

The fields (27)-(28) for a Hertzian electric dipole are typically deduced from the Lorenz-gauge potentials[23] (here given in Gaussian units),

$$\mathbf{A}^{(\mathrm{L})} = \mathrm{Re}\left(-ikp\,\hat{\mathbf{p}}\,\frac{e^{i(kr-\omega t)}}{r}\right) = kp\,\hat{\mathbf{p}}\frac{\sin(kr-\omega t)}{r}\,, \tag{51}$$

$$V^{(\mathrm{L})} = \mathrm{Re}\left(-\frac{i}{k}\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})}\right) = p(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\left(\frac{\cos(kr-\omega t)}{r^2} + k\frac{\sin(kr-\omega t)}{r}\right), \tag{52}$$

recalling the Lorenz-gauge condition, $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{L})} = -\partial V^{(\mathrm{L})}/\partial ct = \mathrm{Re}(ikV^{(\mathrm{L})})$.

In the Coulomb gauge the scalar potential is the instantaneous Coulomb potential,

$$V^{(\mathrm{C})} = \mathrm{Re}\left(p(\hat{\mathbf{p}}\cdot\hat{\mathbf{r^2}})\,\frac{e^{-i\omega t)}}{r}\right) = p(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\frac{\cos\omega t}{r^2}\,. \tag{53}$$

The Coulomb-gauge vector potential is most readily obtained from the relation,

$$\mathbf{E} = \mathrm{Re}\left(-\boldsymbol{\nabla}V^{(\mathrm{C})} - \frac{\partial\mathbf{A}^{(\mathrm{C})}}{\partial ct}\right) = \mathrm{Re}(-\boldsymbol{\nabla}V^{(\mathrm{C})} + ik\mathbf{A}^{(\mathrm{C})}), \tag{54}$$

$$\begin{aligned}
\mathbf{A}^{(\mathrm{C})} &= \mathrm{Re}\left(-\frac{i}{k}\mathbf{E} - \frac{i}{k}\boldsymbol{\nabla}V^{(\mathrm{C})}\right) \\
&= \mathrm{Re}\left(-ikp\,\frac{e^{i(kr-\omega t)}}{r}\hat{\mathbf{r}}\times(\hat{\mathbf{p}}\times\hat{\mathbf{r}}) + p\left[\frac{e^{i(kr-\omega t)}}{r^2} + \frac{i(e^{i(kr-\omega t)} - e^{-i\omega t)}}{kr^3}\right][\hat{\mathbf{p}} - 3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}]\right) \\
&= kp\frac{\sin(kr-\omega t)}{r}[\hat{\mathbf{p}} - (\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}] + p\left[\frac{\cos(kr-\omega t)}{r^2} - \frac{(\sin(kr-\omega t) + \sin\omega t)}{kr^3}\right][\hat{\mathbf{p}} - 3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}].
\end{aligned} \tag{55}$$

Note that the very first term in the last line of eq. (55) is the same as the Lorenz-gauge vector potential (51). This illustrates the transformation of Lorenz-gauge potentials to those in other gauges, as discussed in [26, 36]. Note also that the last term in the last line of eq. (55) is instantaneous, whose contribution to the electric field in eq. (54) cancels that of the instantaneous scalar potential.

The rotational part of the electric field is now given by,

$$\begin{aligned}
\mathbf{E}_{\mathrm{rot}} &= -\frac{1}{c}\frac{\partial\mathbf{A}^{(\mathrm{C})}}{\partial t} \\
&= k^2p\frac{\cos(kr-\omega t)}{r}[\hat{\mathbf{p}} - (\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}] - p\left[\frac{k\sin(kr-\omega t)}{r^2} + \frac{(\cos(kr-\omega t) - \cos\omega t)}{r^3}\right][\hat{\mathbf{p}} - 3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}],
\end{aligned} \tag{56}$$

in agreement with eq. (30).

[23] See, for example, sec. 9.3 of [22] or prob. 8.2 of [35].

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