The Helical Wiggler
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1 Problem

A variant on the electro- or magnetostatic boundary value problem arises in accelerator physics, where a specified field, say \( B(0, 0, z) \), is desired along the \( z \) axis. In general, there exist static fields \( B(x, y, z) \) that reduce to the desired field on the axis, but the “boundary condition” \( B(0, 0, z) \) is not sufficient to insure a unique solution.\(^1\)

For example, find a field \( B(x, y, z) \) that reduces to,

\[
B(0, 0, z) = B_0 \cos k z \hat{x} + B_0 \sin k z \hat{y} \tag{1}
\]
on the \( z \) axis. In this, the magnetic field rotates around the \( z \) axis as \( z \) advances.

Show how the use of rectangular or cylindrical coordinates leads “naturally” to different forms for \( B \).

One 3-dimensional field extension of (1) is the so-called helical wiggler \([2, 3]\), which obeys the auxiliary requirement that the field at \( z + \delta \) be the same as the field at \( z \), but rotated by angle \( k \delta \). Show that this field pattern can be realized by a current-carrying wire that is wound in a helix of period \( \lambda = 2\pi/k \) [4].

2 Solution

2.1 Solution in Rectangular Coordinates

We first seek a solution in rectangular coordinates, and expect that separation of variables will apply. Thus, we consider the form,

\[
\begin{align*}
B_x &= f(x)g(y) \cos kz, \tag{2} \\
B_y &= F(x)G(y) \sin kz, \tag{3} \\
B_z &= A(x)B(y)C(z). \tag{4}
\end{align*}
\]

Then,

\[
\nabla \cdot B = 0 = f'g \cos k z + FG' \sin k z + ABC', \tag{5}
\]

where the ′ indicates differentiation of a function with respect to its argument. Equation (5) can be integrated with respect to \( z \) to give,

\[
ABC = -\frac{f'g}{k} \sin k z + \frac{FG'}{k} \cos k z. \tag{6}
\]

\(^1\)If the axial field has only an axial component a unique solution obtains [1].
The z component of $\nabla \times \mathbf{B} = 0$ tells us that,
\[
\frac{\partial B_z}{\partial y} = f' \cos k z = \frac{\partial B_y}{\partial x} = F' G \sin k z.
\]  
(7)

For this to hold at all $x$ and $y$ we must have $g' = 0 = F'$, which implies that $g$ and $F$ are constant, say 1. Likewise,
\[
\frac{\partial B_x}{\partial z} = - f k \sin k z = \frac{\partial B_z}{\partial y} = A' B C = - \frac{f''}{k} \sin k z,
\]  
(8)

using eqs. (6)-(7). Thus, $f'' - k^2 f = 0$, so,
\[
f = f_1 e^{kx} + f_2 e^{-kx}.
\]  
(9)

Finally,
\[
\frac{\partial B_y}{\partial z} = G k \cos k z = \frac{\partial B_y}{\partial y} = A'B'C = \frac{G''}{k} \sin k z,
\]  
(10)

so,
\[
G = G_1 e^{ky} + G_2 e^{-ky}.
\]  
(11)

The “boundary conditions” $f(0) = B_0 = G(0)$ are satisfied by,
\[
f = B_0 \cosh kx, \quad G = B_0 \cosh ky
\]  
(12)

which together with eq. (6) leads to the solution,
\[
B_x = B_0 \cosh kx \cos k z,
\]  
(13)

\[
B_y = B_0 \cosh ky \sin k z,
\]  
(14)

\[
B_z = - B_0 \sinh kx \sin k z + B_0 \sinh ky \cos k z,
\]  
(15)

This satisfies the last “boundary condition” that $B_z(0, 0, z) = 0$.

However, this solution does not have helical symmetry.

### 2.2 Solution in Cylindrical Coordinates

Suppose instead, we look for a solution in cylindrical coordinates $(r, \theta, z)$. We again expect separation of variables, but we seek to enforce the helical symmetry that the field at $z + \delta$ be the same as the field at $z$, but rotated by angle $k \delta$. This symmetry implies that the argument $kz$ should be replaced by $kz - \theta$, and that the field has no other $\theta$ dependence.

We begin constructing our solution with the hypothesis that,
\[
B_r = F(r) \cos(kz - \theta),
\]  
(16)

\[
B_\theta = G(r) \sin(kz - \theta).
\]  
(17)

To satisfy the condition (1) on the $z$ axis, we first transform this to rectangular components,
\[
B_z = F(r) \cos(kz - \theta) \cos \theta + G(r) \sin(kz - \theta) \sin \theta,
\]  
(18)

\[
B_y = - F(r) \cos(kz - \theta) \cos \theta - G(r) \sin(kz - \theta) \cos \theta,
\]  
(19)
from which we learn that the “boundary conditions” on $F$ and $G$ are

$$F(0) = G(0) = B_0. \quad (20)$$

A suitable form for $B_z$ can be obtained from $(\nabla \times B)_r = 0$:

$$\frac{1}{r} \frac{\partial B_z}{\partial \theta} = \frac{\partial B_\theta}{\partial z} = kG \cos(kz - \theta), \quad (21)$$

so,

$$B_z = -krG \sin(kz - \theta), \quad (22)$$

which vanishes on the $z$ axis as desired.

From either $(\nabla \times B)_\theta = 0$ or $(\nabla \times B)_z = 0$ we find that,

$$F = \frac{d(rG)}{dr} = \frac{d(krG)}{dkr}. \quad (23)$$

Then, $\nabla \cdot B = 0$ leads to,

$$(kr)^2 \frac{d^2 (krG)}{d(kr)^2} + kr \frac{d(krG)}{d(kr)} - [1 + (kr)^2](krG) = 0. \quad (24)$$

This is the differential equation for the modified Bessel function of order 1 [5]. Hence,

$$G = C \frac{I_1(kr)}{kr} = C \left[ 1 + \frac{(kr)^2}{8} + \cdots \right], \quad (25)$$

$$F = C \frac{dI_1}{d(kr)} = C \left( I_0 - \frac{I_1}{kr} \right) = C \left[ 1 + \frac{3(kr)^2}{8} + \cdots \right]. \quad (26)$$

The “boundary conditions” (20) require that $C = 2B_0$, so our second solution is,

$$B_r = 2B_0 \left( I_0(kr) - \frac{I_1(kr)}{kr} \right) \cos(kz - \theta), \quad (27)$$

$$B_\theta = 2B_0 \frac{I_1}{kr} \sin(kz - \theta), \quad (28)$$

$$B_z = -2B_0 I_1(kr) \sin(kz - \theta), \quad (29)$$

which is the form discussed in [3].

### 2.3 Magnetic Field Due to a Double Helix

This section follows [6].

We consider a wire that carries current $I$ and is wound in the form of a helix of radius $a$ and period $\lambda = 2\pi/k$. A suitable equation of this helix is,

$$x_1 = a \sin k\lambda, \quad y_1 = -a \cos k\lambda. \quad (30)$$
The magnetic field due to this winding has a nonzero $z$ component along the axis, which is not desired. Therefore, we also consider a second helical winding,

$$x_2 = -a \sin k z, \quad y_2 = a \cos k z,$$

(31)

which is offset from the first by half a period and which carries current $-I$. The combined magnetic field from the two helices has no component along their common axis.

The magnetic field $B$ where we made the substitution $z = s \sin u$ also satisfies eq. (24). We integrate the last integral by parts, using,

$$\int \frac{dz}{\sqrt{\lambda^2 + (2\pi a)^2}} = d\zeta (\pm ka \cos k z', \pm ka \sin k z', 1).$$

(34)

The magnetic field $\mathbf{B}$ at a point $\mathbf{r} = (0, 0, z)$ on the axis is given by,

$$B(0, 0, z) = \frac{I}{c} \int_1 \frac{d\mathbf{l}_1 \times (\mathbf{r}'_1 - \mathbf{r})}{|\mathbf{r}'_1 - \mathbf{r}|^3} - \frac{I}{c} \int_2 \frac{d\mathbf{l}_2 \times (\mathbf{r}'_2 - \mathbf{r})}{|\mathbf{r}'_2 - \mathbf{r}|^3}$$

$$= 2Ia \int_{-\infty}^{\infty} \frac{dz'}{(2\pi a)^3/2 \sqrt{\lambda^2 + (2\pi a)^2}} \left[ \hat{\mathbf{x}} (k(z' - z) \sin k z' + \cos k z') + \hat{\mathbf{y}} (-k(z' - z) \cos k z' + \sin k z') \right]$$

$$= 2Ic \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{3/2}} \left[ \hat{\mathbf{x}} (kat \sin(kat + k z) + \cos(kat + k z)) + \hat{\mathbf{y}} (-kat \cos(kat + k z) + \sin(kat + k z)) \right]$$

$$= \frac{4I}{c} \left( \hat{\mathbf{x}} \cos k z + \hat{\mathbf{y}} \sin k z \right) \left[ \frac{1}{ka} \int_0^{\infty} \frac{\cos kat}{(1 + t^2)^{3/2}} dt + \int_0^{\infty} \frac{t \sin kat}{(1 + t^2)^{3/2}} dt \right],$$

(35)

where we made the substitution $z' - z = at$ in going from the second line to the third. Equation 9.6.25 of [5] tells us that,

$$\int_0^{\infty} \frac{\cos kat}{(1 + t^2)^{3/2}} dt = ka K_1(ka),$$

(36)

where $K_1$ also satisfies eq. (24). We integrate the last integral by parts, using,

$$u = \sin kat, \quad dv = \frac{t \, dt}{(1 + t^2)^{3/2}}, \quad \text{so} \quad du = ka \cos kat \, dt, \quad v = -\frac{1}{\sqrt{1 + t^2}}.$$  

(37)

Thus,

$$\int_0^{\infty} \frac{t \sin kat}{(1 + t^2)^{3/2}} dt = ka \int_0^{\infty} \frac{\cos kat}{\sqrt{1 + t^2}} dt = ka K_0(ka),$$

(38)
using 9.6.21 of [5]. Hence,

$$\mathbf{B}(0, 0, z) = \frac{4I_k}{c} \left[ k a K_0(ka) + K_1(ka) \right] (\hat{x} \cos k z + \hat{y} \sin k z). \quad (39)$$

Both $K_0(ka)$ and $K_1(ka)$ have magnitudes $\approx 0.5e^{-ka}$ for $ka \approx 1$. That is, the field on the axis of the double helix is exponentially damped in the radius $a$ for a fixed current $I$.

References


