Heat Flow from a Point Source at the End of a Bar

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1 Problem

As a simple example of 3-dimensional heat flow, deduce the steady-state temperature distribution inside a semi-infinite square bar with a point source of heat somewhere on its square face, assuming no heat flow across other surfaces (except the square face at infinity).\(^1\)

2 Solution

The heat flux vector \( \mathbf{J} \) obeys,

\[
\mathbf{J} = -\kappa \nabla T,
\]

where \( \kappa \) is the thermal conductivity and \( T \) is the temperature distribution. Energy is conserved in the interior of the bar, so in a steady state \( \nabla \cdot \mathbf{J} = 0 \) there, and hence \( \nabla^2 T = 0 \).

We consider a separation-of-variable solution in a rectangular coordinate system, taking the heat source \( Q \) to be at \((x_0,y_0,0)\), with the bar extending over the \( z \geq 0 \) with square cross section \(|x|,|y| \leq a/2\). The normal derivative of the temperature is zero at the surfaces across which no heat flows, so the boundary conditions are,\(^2\)

\[
\begin{align*}
\frac{\partial T(x,y,0)}{\partial z} &= -\frac{Q}{\kappa} \delta(x-x_0, y-y_0), \\
\frac{\partial T(0,y,z)}{\partial x} &= \frac{\partial T(a,y,z)}{\partial x} = 0 = \frac{\partial T(x,0,z)}{\partial y} = \frac{\partial T(x,a,z)}{\partial y}. 
\end{align*}
\]

A separated form that obeys \( \nabla^2 T = 0 \) and satisfies condition (3) is,\(^3\)

\[
T = \sum_{m,n=0}^{\infty} C_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2} \pi z/a} - Az. 
\]

Condition (2) is then,

\[
A + \frac{2\pi}{a} \sum_{m,n=0}^{\infty} \sqrt{m^2+n^2} C_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} = \frac{Q}{\kappa} \delta(x-x_0, y-y_0). 
\]

\(^1\)Equivalently, consider a square bar of infinite length with a point source somewhere inside.
\(^2\)It seems not possible to obtain an analytic solution for a bar of finite length with a point source on one end and the other end at fixed temperature, if no heat flows across its other surfaces.
\(^3\)This type of solution may have been first given by Fourier, sec. 321 of [1].
On multiplying by \( \cos \frac{2k\pi x}{a} \cos \frac{2n\pi y}{a} \) and integrating over the area of the square cross section of the bar, we find that,

\[
A = \frac{Q}{a^2 \kappa}, \quad C_{k,l} = \frac{2Q}{\pi a \kappa} \begin{cases} 
\text{undefined} & (k = l = 0), \\
\frac{1}{2} \cos \frac{l\pi y}{a} & (k = 0, \ l \geq 1), \\
\frac{1}{2} \cos \frac{k\pi x}{a} & (l = 0, \ k \geq 1), \\
\sqrt{\frac{2}{\kappa^2 + \ell^2}} \cos \frac{k\pi x}{a} \cos \frac{l\pi y}{a} & (k, \ l \geq 1).
\end{cases}
\]

(6)

Hence, on redefining the undetermined constant \( C_{00} \) as \( T_0 \),

\[
T = T_0 - \frac{Qz}{a^2 \kappa} + \frac{2Q}{\pi a \kappa} \left( \sum_{m=1}^\infty \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} e^{-2m\pi z/a} + \sum_{n=1}^\infty \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2n\pi z/a} \right)
\]

\[
+ 2 \sum_{m,n=1}^\infty \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2\sqrt{m^2 + n^2} \pi z/a} \right) .
\]

(7)

and the heat-flow vector (1) has components,

\[
J_x = \frac{2Q}{a^2} \left( \sum_{m=1}^\infty \sin \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} e^{-2m\pi z/a} \right)
\]

\[
+ 2 \sum_{m,n=1}^\infty \sin \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \sin \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2\sqrt{m^2 + n^2} \pi z/a} \right) .
\]

(8)

\[
J_y = \frac{2Q}{a^2} \left( \sum_{n=1}^\infty \sin \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2n\pi z/a} \right)
\]

\[
+ 2 \sum_{m,n=1}^\infty \sin \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \sin \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2\sqrt{m^2 + n^2} \pi z/a} \right) .
\]

(9)

\[
J_z = \frac{Q}{a^2} \left( 1 + 2 \sum_{m=1}^\infty \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^\infty \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2n\pi z/a} \right)
\]

\[
+ 4 \sum_{m,n=1}^\infty \cos \frac{2m\pi x}{a} \cos \frac{2m\pi x_0}{a} \cos \frac{2n\pi y}{a} \cos \frac{2n\pi y_0}{a} e^{-2\sqrt{m^2 + n^2} \pi z/a} \right) .
\]

(10)

For the particular case that the point source is at the center of the end face, \( x_0 = y_0 = 0 \),

\[
T = T_0 - \frac{Qz}{a^2 \kappa} + \frac{Q}{\pi a \kappa} \left( \sum_{m=1}^\infty \cos \frac{2m\pi x}{a} e^{-2m\pi z/a} + \sum_{n=1}^\infty \cos \frac{2n\pi y}{a} e^{-2n\pi z/a} \right)
\]

\[
+ 2 \sum_{m,n=1}^\infty \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2 + n^2} \pi z/a} \right) .
\]

(11)

and the heat-flow vector (1) has components,
\[ J_x = \frac{2Q}{a^2} \left( \sum_{m=1}^{\infty} \sin \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{m,n=1}^{\infty} m \sin \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right), \] (12)

\[ J_y = \frac{2Q}{a^2} \left( \sum_{n=1}^{\infty} \sin \frac{2n\pi y}{a} e^{-2n\pi z/a} + 2 \sum_{m,n=1}^{\infty} n \cos \frac{2m\pi x}{a} \sin \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right), \] (13)

\[ J_z = \frac{Q}{a^2} \left( 1 + 2 \sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^{\infty} \cos \frac{2n\pi y}{a} e^{-2n\pi z/a} + 4 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right). \] (14)

The figures below\(^4\) shows the lines of the heat-flow vector \( \mathbf{J} \) in the midplane \( y = 0 \),

\[ J_x(x,0,z) = \frac{2Q}{a^2} \left( \sum_{m=1}^{\infty} \sin \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{m,n=1}^{\infty} m \sin \frac{2m\pi x}{a} \frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}} \right), \] (15)

\[ J_y(x,0,z) = 0, \] (16)

\[ J_z(x,0,z) = \frac{Q}{a^2} \left( 1 + 2 \sum_{m=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2m\pi z/a} + 2 \sum_{n=1}^{\infty} e^{-2n\pi z/a} + 4 \sum_{m,n=1}^{\infty} \cos \frac{2m\pi x}{a} e^{-2\sqrt{m^2+n^2}\pi z/a} \right). \] (17)

Indices \( m \) and \( n \) were evaluated up to 1 in the left figure and to 20 in the right; indices higher than 1 mainly affect the region close to \( z = 0 \) where the delta-function boundary condition (5) is being approximated.

\(^4\)The figures were generated via the Mathematica notebook http://kirkmcd.princeton.edu/examples/heatflow.nb.
The figure indicates that the heat flow is essentially parallel to the $z$-axis for $z \gtrsim a/2$ from the point source, which is agreeable with naïve expectations.\footnote{This example does not appear in the great compendium [2] of lore on heat conduction, although the ingredients of the solution are, of course, well represented there. For example, sec. 14.3-III gives a 2-dimensional version of the present problem.}

**References**

http://kirkmcd.princeton.edu/examples/statmech/fourier_22_english.pdf