1 Problem

Estimate the period $\tau$ of a “simple” harmonic oscillator consisting of a zero-rest-length massless spring of constant $k$ that is connected to a rest mass $m_0$ (with the other end of the spring fixed to the origin), taking in account the relativistic mass.

2 Solution

2.1 Quick Estimates

Ignoring relativistic effects, the angular frequency $\omega_0$ and the period $\tau_0$ of the oscillator are,

$$\omega_0 = \sqrt{\frac{k}{m_0}}, \quad \tau_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m_0}{k}}. \quad (1)$$

In this approximation, the oscillating mass has position and velocity,

$$x = A \cos \omega_0 t, \quad v = -A \omega_0 \sin \omega_0 t. \quad (2)$$

In general, the oscillating mass has (time-dependent) relativistic mass,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0 \left(1 + \frac{v^2}{2c^2}\right), \quad (3)$$

where $c$ is the speed of light in vacuum. We expect that the period $\tau$ of oscillation of the relativistic mass can be approximated as,

$$\tau \approx \frac{2\pi}{\langle m \rangle} = 2\pi \sqrt{\frac{\langle m \rangle}{k}} > \tau_0, \quad (4)$$

where $\langle m \rangle > m_0$ is an appropriate average of the relativistic mass. This might be the time average,

$$\langle m \rangle_t = \frac{1}{\tau} \int_0^\tau m(t) \, dt \approx \frac{m_0}{\tau} \int_0^\tau \left(1 + \frac{v^2}{2c^2}\right) \, dt \approx m_0 \left(1 + \frac{1}{2\tau_0 c^2} \int_0^{\tau_0} A^2 \omega_0^2 \cos^2 \omega_0 t \, dt\right)$$

$$= m_0 \left(1 + \frac{A^2 \omega_0^2}{4c^2}\right) = m_0 \left(1 + \frac{kA^2}{4m_0 c^2}\right), \quad (5)$$

in which case,

$$\tau \approx \tau_0 \sqrt{1 + \frac{kA^2}{4m_0 c^2}} \approx \tau_0 \left(1 + \frac{kA^2}{8m_0 c^2}\right), \quad \langle m \rangle = \langle m \rangle_t. \quad (6)$$
However, it could be that the spatial average is more appropriate,

\[ \langle m \rangle_x = \frac{1}{A} \int_0^A m(x) \, dx \approx \frac{m_0}{A} \int_0^A \left( 1 + \frac{v^2}{2c^2} \right) \, dx \approx m_0 \left[ 1 + \frac{1}{2Ac^2} \int_0^A A^2 \omega_0^2 \left( 1 - \frac{x^2}{A^2} \right) \, dx \right] \]

noting that \( \sin \omega t = \sqrt{1 - \cos^2 \omega t} \approx \sqrt{1 - x^2/A^2} \), in the approximation that oscillating mass has \( x \)-coordinate \( x = A \cos \omega t \). In this case,

\[ \tau \approx \tau_0 \sqrt{1 + \frac{kA^2}{3m_0c^2}} \approx \tau_0 \left( 1 + \frac{kA^2}{6m_0c^2} \right), \quad \langle m \rangle = \langle m \rangle_x. \tag{8} \]

As many other averages of the relativistic mass can be imagined, we seek a method that clarifies what type of approximation is best.

### 2.2 A Better Estimate

A different approach is to note that the motion is periodic with spatial amplitude \( A \), and so the period \( \tau \) can be computed as,

\[ \tau = 4 \int_0^A \frac{dt}{dx} \, dx = 4 \int_0^A \frac{dx}{\sqrt{1 - v^2/c^2}}. \tag{9} \]

Total energy \( E \) is conserved in this example,

\[ E = mc^2 + \frac{kx^2}{2} = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} + \frac{kx^2}{2} = m_0c^2 + \frac{kA^2}{2}, \tag{10} \]

where the potential energy of the system is \( kx^2/2 \), such that,\(^1\)

\[ \frac{1}{v} = \frac{\tau_0}{2\pi} \frac{1 + k(A^2 - x^2)/2m_0c^2}{\sqrt{A^2 - x^2}/\sqrt{1 + k(A^2 - x^2)/4m_0c^2}} \approx \frac{\tau_0}{2\pi} \left( \frac{1}{\sqrt{A^2 - x^2}} + \frac{3k\sqrt{A^2 - x^2}}{8m_0c^2} \right). \tag{11} \]

Hence,

\[ \tau \approx \frac{2\tau_0}{\pi} \left( \int_0^A \frac{dx}{\sqrt{A^2 - x^2}} + \frac{3k}{8m_0c^2} \int_0^A \sqrt{A^2 - x^2} \, dx \right) = \tau_0 \left( 1 + \frac{3kA^2}{16m_0c^2} \right). \tag{12} \]

The correction term in this result is 2% larger than that in the estimate (8) based on the spatial average of the relativistic mass.

The “exact” period of a relativistic harmonic oscillator can be given as an elliptic integral. A series approximation to this integral is given in [2].

\(^1\)There is a sign error in the correction term of eq. (7-150), p. 325 of [1], which corresponds to eq. (11) of the present note. Thanks to Bill Jones for pointing this out.
References
