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Haldane's Cart

D. Haldane has posed the following problem. A cart of mass m is free to roll on a horizontal surface. The cart contains water whose mass $M(t)$ varies due to a drain located in the bottom of the cart at a distance a from the center of the cart. Describe the motion of the cart.

Solution

We let x be the coordinate of the center of the cart, and suppose that the cart starts from rest at the origin at $t = 0$. We will consider several cases:

I. The water leaves the cart with zero horizontal velocity in the lab frame.

II. The water leaves the cart with zero horizontal velocity in the rest frame of the cart. This is believed to be more realistic.

A. The mass of the water remaining in the cart obeys $M = M_0 - Rt$.

B. The mass of the water remaining in the cart obeys $M = M_0 e^{-kt}$.

C. The mass of the water remaining in the cart obeys $M = (\sqrt{M_0} - St)^2$. By conservation of energy, the velocity v of the flow out the drain is given by $v^2 = 2gh \propto$ (height of water in the cart) $\propto$ (mass of water left in the cart). Then $\dot{M} \propto v \propto \sqrt{M}$. This case is the most realistic.

We will also consider the limit that the mass of the cart vanishes.

The problem is tricky in that the water in the cart is moving relative to the cart. We present a solution that avoids the need for a detailed discussion of this motion.

Since there are no external horizontal forces on the system of cart + water (including the water that has drained out), the c.m. of the whole system must remain fixed at the origin. Thus at time t,

$$0 = (m + M(t))x(t) + \text{ c.m. coord of all water drained out up to time } t.$$

In the interval dt' at an ealier time t', mass dM of water drains out with horizontal velocity $u(t')$. [We will take $dM < 0$ by convention so that $\dot{M} = dM/dt < 0$ describes the rate of change of the mass of the water remaining in the cart.] At time t' the drain was at $x(t') + a$, so at time t the element dM is at $x(t') + a + u(t')(t - t')$.

Then the c.m. of the whole system obeys

$$0 = (m + M(t))x(t) + \int_0^t dt'(-\dot{M}(t'))[x(t') + a + u(t')(t - t')].$$

$$= (m + M(t))x(t) - \int_0^t dt'\dot{M}(t')[x(t') + a - t'u(t')] - t\int_0^t dt'\dot{M}(t')u(t').$$

We take time derivatives of this to find the equation of motion:

$$0 = (m + M)\dot{x} - \dot{M}a - \int_0^t dt'\dot{M}(t')u(t'). \tag{1}$$

On taking the derivative again we find:

$$0 = (m + M)\ddot{x} + \dot{M}\dot{x} - \dot{M}u - \ddot{M}a. \tag{2}$$

I have also derived this equation of motion using a model of how the water flows inside the cart, and keeping track of momentum rather than c.m. position.

Case I: $u = 0$.

In this case no horizontal momentum leaves the cart, and hence the total momentum of the cart + water inside always remains zero. Nonetheless, we find that the cart moves in the $-x$ direction.

From eq. (1) above we have

$$\dot{x} = \frac{a\dot{M}}{m + M}.$$

Note that

$$\dot{x}(0) \equiv v_0 = \frac{a\dot{M}(0)}{m + M_0} < 0.$$

This analysis is not detailed enough to consider the transient when the drain first opens and the water starts moving inside the cart. This transient gives the cart the initial velocity v_0.

A. Suppose that $M = M_0 - Rt$.

Then

$$\dot{x} = -\frac{aR}{m + M_0 - Rt},$$

and

$$x = a\ln\frac{m + M_0 - Rt}{m + M_0}.$$

In the limit of a massless cart ($m = 0$), the cart ends up at time $t = M_0/R$ at $x = -\infty$ with $\dot{x} = -\infty$. This is no contradiction: the center of mass of the water that has drained out can be readily shown to still be zero.

B. Suppose that $M = M_0 e^{-kt}$.
Then

$$\dot{x} = -\frac{akM_0}{M_0 + me^{kt}},$$

and

$$x = -a\left(kt - \ln\frac{M_0 + me^{kt}}{M_0 + m}\right),$$

using Gradshteyn and Ryzhik 2.313.1. As $t \to \infty$, we have $x \to -a\ln(1 + M_0/m)$ exactly as in case A. The general character of the motion is as in case A.

At $t = 0^+$ the velocity is $v_0 = -akM_0/(M_0 + m)$.

C. Suppose that $M(t) = (\sqrt{M_0} - St)^2$.
Then

$$\dot{x} = -\frac{2aS(\sqrt{M_0} - St)}{m + (\sqrt{M_0} - St)^2},$$

which is always < 0, but goes to zero as the cart empties.

$$x = a\ln\frac{m + (\sqrt{M_0} - St)^2}{m + M_0} + 2a\sqrt{\frac{M_0}{m}}\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right).$$

The water has all drained out at time $t = \sqrt{M_0}/S$, at which time the cart has position

$$x = a\ln\frac{m}{m + M_0} + 2a\sqrt{\frac{M_0}{m}}\tan^{-1}\sqrt{\frac{M_0}{m}}.$$

This is always < 0.

Case II: $u = \dot{x}$.

In this case the falling water takes on the instantaneous horizontal velocity of the cart, and hence imparts a reaction force on the cart that opposes its motion.

From eq. (2) above we have

$$\ddot{x} = \frac{a\ddot{M}}{m + M}.$$

A. Suppose $M = M_0 - Rt$.

Then $\ddot{M} = 0$ and $\dot{x}$ is constant at its initial value v_0. I presume the value of v_0 is just $\dot{x}(0) = -aR/(m + M_0)$ found in case IA above. When the water has all drained out at time $t + M_0/R$, the cart is at $x = -aM_0/(m + M_0)$, and is moving in the $-x$ direction. A massless cart will have moved distance a at the moment it becomes empty.

B. Suppose $M = M_0 e^{-kt}$.

Then

$$\ddot{x} = \frac{ak^2 M_0}{M_0 + me^{kt}},$$

The acceleration is in the $+x$ direction, and might be sufficient to reverse the initial velocity in the $-x$ direction.

$$\dot{x} = v_0 + ak\left(kt - \ln\frac{M_0 + me^{kt}}{M_0 + m}\right),$$

and

$$x = v_0 t + \frac{ak^2t^2}{2} - ak\int_0^t dt \ln\frac{M_0 + me^{kt}}{M_0 + m}.$$

The latter integral cannot be evaluated in closed form (according to Gradshteyn and Ryzhik 2.727.2 and 2.728.2).

I presume that v_0 has the value found in case IB above. Then for large t,

$$\dot{x} \to v_0 - ak\ln\frac{m}{M_0 + m} = -ak\left(\frac{M_0}{M_0 + m} + \ln\frac{m}{M_0 + m}\right).$$

This is always positive, although for large cart masses we have $\dot{x} \to (ak/2)(M_0/m)^2$ which approaches zero. Also in the limit of large cart masses, the cart is always at $x < 0$, but for small enough cart masses the cart could end up with $x > 0$.

Consider the limit of a massless cart ($m \to 0$). The equation of motion is just $\ddot{x} = ak^2$, so

$$\dot{x} = v_0 + ak^2 t = -ak(1 - kt),$$

using $v_0 = -ak$ in this limit. For $t > 1/k$ the velocity of the cart is positive, and opposite to the initial direction of the motion.

$$x = v_0 t + \frac{ak^2t^2}{2} = -akt(1 - kt/2).$$

The cart returns to the origin at $t = 2/k$ and moves off to $x = +\infty$ thereafter.

Left open is the question of for what value of M_0/m does the cart return exactly to the origin at large times.

C. Suppose $M(t) = (\sqrt{M_0} - St)^2$.
Then

$$\ddot{x} = \frac{2aS^2}{m + (\sqrt{M_0} - St)^2},$$

which is always positive, so may be able to reverse the initial negative velocity.

$$\dot{x} = 2aS\left[-\frac{\sqrt{M_0}}{m + m_0} + \frac{1}{\sqrt{m}}\left(\tan^{-1}\sqrt{\frac{M_0}{m}} - \tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right)\right],$$

using $v_0 = -2aS\sqrt{M_0}/(m + M_0)$ from case IC above. At large times this becomes

$$\dot{x} \to 2aS\left[-\frac{\sqrt{M_0}}{m + m_0} + \frac{1}{\sqrt{m}}\tan^{-1}\sqrt{\frac{M_0}{m}}\right].$$

I believe that this is always positive, but for large cart masses m, $\dot{x}(\infty) \to (4/3)aSM_0^{3/2}/m^2$, which approaches zero.
We can integrate to get

$$x = 2aSt\left[-\frac{\sqrt{M_0}}{m + m_0} + \frac{1}{\sqrt{m}}\tan^{-1}\sqrt{\frac{M_0}{m}}\right] - 2a\left[\frac{1}{2}\ln\frac{m + (\sqrt{M_0} - St)^2}{m + M_0} + \right.$$

$$\left. + \sqrt{\frac{M_0}{m}}\tan^{-1}\sqrt{\frac{M_0}{m}} - \frac{\sqrt{M_0} - St}{\sqrt{m}}\tan^{-1}\frac{\sqrt{M_0} - St}{\sqrt{m}}\right].$$

When the cart is empty at time $t = \sqrt{M_0}/S$ its position is

$$x_{\text{empty}} = a\left(\ln\frac{m + M_0}{m} - 2\frac{M_0}{m + M_0}\right).$$

The critical value of the mass of the cart that just returns to the origin when empty is $m = 1.35M_0$.

We also consider the limit of a massles cart. Then

$$\ddot{x} = \frac{2aS^2}{(\sqrt{M_0} - St)^2},$$

$$\dot{x} = -\frac{4aS}{\sqrt{M_0}} + \frac{2aS}{\sqrt{M_0} - St},$$

using $v_0 = -2aS/\sqrt{M_0}$, and

$$x = -\frac{4aSt}{\sqrt{M_0}} + 2a\ln\frac{\sqrt{M_0}}{\sqrt{M_0} - St}.$$

Here the cart reverses its direction at time $t = \sqrt{M_0}/2S$ (*i.e.*, when it is still 3/4 full), passes the origin at some later time and moves toward large x as it empties.