Emittance Growth
from Weak Relativistic Effects

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1 Problem

Although phase volume is invariant under canonical transformations of a Hamiltonian system (see, for example, [1]), approximations to phase volume such as rms emittance are not. Deduce approximate expressions for the growth of the various 2-D rms emittances of a Gaussian bunch of particles of mass $m$ and charge $q$, initially centered on the origin and with $\langle p_z \rangle = 0 = \langle p_y \rangle$ but with nonzero energy spread about a central energy $E_0$, as this bunch propagates in a region of zero electromagnetic field. For the longitudinal emittance, consider both coordinates ($z, p_z$) and ($t, p_t = -E_{total}$) [1].

2 Solution

This solution is an extension of [2, 3, 4]. Numerical examples of rms emittance growth during propagation of a “beam” in a field-free region are given on slide 8 of [5].

2.1 Emittances when $t$ Is the Independent Variable

When using time $t$ as the independent variable the canonical coordinates are $x, y, z, p_x, p_y, p_z$ and the initial conditions are at time $t = 0 \equiv t_0$. We suppose the initial bunch is Gaussian, with nonzero first and second moments,

$$\langle p_z(t = 0) \rangle \equiv \langle p_z(0_t) \rangle = p_{0t, z}, \quad E_{0t} \equiv c\sqrt{m^2 c^2 + p_{0t, z}^2} < \langle E(0) \rangle,$$

$$\langle x^2(0_t) \rangle = \langle y^2(0_t) \rangle = \sigma^2_{1z}, \quad \langle z^2(0_t) \rangle = \sigma^2_{zz}, \quad \langle p^2_x(0_t) \rangle = \langle p^2_y(0_t) \rangle = \sigma^2_{p_{1z}},$$

$$\langle p^2_z(0_t) - p^2_{0t, z} \rangle = \langle (p_z(0_t) - p_{0t, z})^2 \rangle = \sigma^2_{p_{zt}}, \quad \langle p^2_z(0_t) \rangle = p^2_{0t, z} + \sigma^2_{p_{zt}}, \quad (1)$$

That is, there are no cross correlations initially. We will also need to know some higher moments, such as,

$$\langle p^4_z(0_t) \rangle = \langle p^4_y(0_t) \rangle = 3\sigma^4_{p_{1z}}, \quad \langle p^6_y(0_t) \rangle = \langle p^6_y(0_t) \rangle = 15\sigma^6_{p_{1z}},$$

$$\langle (p_z(0_t) - p_{0t, z})^6 \rangle = 0, \quad \langle p^3_z(0_t) \rangle = p^3_{0t, z} + 3\sigma^2_{p_{zt}}p_{0t, z},$$

$$\langle (p_z(0_t) - p_{0t, z})^4 \rangle = 3\sigma^4_{p_{zt}}, \quad \langle p^4_z(0_t) \rangle = p^4_{0t, z} + 6\sigma^2_{p_{zt}}p_{0t, z} + 3\sigma^4_{p_{zt}},$$

$$\langle p^2_z(0_t) - p^2_{0t, z} \rangle = 5\sigma^2_{p_{zt}}p^2_{0t, z} + 3\sigma^4_{p_{zt}}, \quad \langle (p^2_z(0_t) - p^2_{0t, z})^2 \rangle = 4\sigma^2_{p_{zt}}p^2_{0t, z} + 3\sigma^4_{p_{zt}},$$

$$\langle (p_z(0_t) - p_{0t, z})^5 \rangle = 0, \quad \langle p^5_z(0_t) \rangle = p^5_{0t, z} + 10\sigma^2_{p_{zt}}p^3_{0t, z} + 15\sigma^4_{p_{zt}}p_{0t, z},$$
It turns out that we need to expand 1 which follow from the assumption that \( p_z(0) - p_{0t,z} \) has a Gaussian distribution.

The equations of motion are,

\[
\begin{align*}
p_x(t) &= p_x(0_t), & p_y(t) &= p_y(0_t), & p_z(t) &= p_z(0_t), \\
v_x(t) &= v_x(0_t), & v_y(t) &= v_y(0_t), & v_z(t) &= v_z(0_t), \\
E(t) &= E(0_t) = c\sqrt{m^2 c^2 + p_x^2(0_t) + p_y^2(0_t) + p_z^2(0_t) - \sigma^2_{z0_t}}, \\
x_i(t) &= x_i(0_t) + v_i(t)t = x_i(0_t) + \frac{c^2t p_i(0_t)}{E(0_t)}, \\
x_i^2(t) &= x_i^2(0_t) + \frac{2c^2t x_i(0_t)p_i(0_t)}{E(0_t)} + \frac{c^4t^2p_i^2(0_t)}{E^2(0_t)}. \quad (3)
\end{align*}
\]

It turns out that we need to expand 1/E(0_t) to order 1/E_0^5 (and 1/E^2(0_t) to order 1/E_0^6),

\[
\begin{align*}
\frac{1}{E(0_t)} &\approx \frac{1}{E_0^5} \left\{ 1 - \frac{c^2}{2E_0^5} (p_x^2(0_t) + p_y^2(0_t) + p_z^2(0_t) - \sigma^2_{z0_t}) \\
&\quad + \frac{3c^4}{8E_0^4} \left[ p_x^4(0_t) + p_y^4(0_t) + (p_z^2(0_t) - \sigma^2_{z0_t})^2 \right] \\
&\quad + 2p_x^2(0_t)p_y^2(0_t) + 2(p_x^2(0_t) + p_y^2(0_t))(p_z^2(0_t) - \sigma^2_{z0_t}) \right\}, \\
\frac{1}{E^2(0_t)} &\approx \frac{1}{E_0^6} \left\{ 1 - \frac{c^2}{E_0^5} (p_x^2(0_t) + p_y^2(0_t) + p_z^2(0_t) - \sigma^2_{z0_t}) \\
&\quad + \frac{c^4}{E_0^4} \left[ p_x^4(0_t) + p_y^4(0_t) + (p_z^2(0_t) - \sigma^2_{z0_t})^2 \right] \\
&\quad + 2p_x^2(0_t)p_y^2(0_t) + 2(p_x^2(0_t) + p_y^2(0_t))(p_z^2(0_t) - \sigma^2_{z0_t}) \right\}. \quad (4)
\end{align*}
\]

Note that we expand in the small quantity \( p_x^2(0_t) - \sigma^2_{z0_t} \) rather than in \( \Delta p_z = p_z(0_t) - p_{0t,z} \).

The nonzero first and second moments in \( x \) and \( y \) at time \( t \) are,

\[
\begin{align*}
\langle p_x^2(t) \rangle &= \langle p_y^2(t) \rangle = \sigma^2_{p_{i\perp}}, \\
\langle x^2(t) \rangle &\approx \sigma^2_{z0_t} + \frac{c^4\sigma^2_{p_{i\perp}} t^2}{E_0^2} \left( 1 - \frac{c^2}{E_0^5} (4\sigma^2_{p_{i\perp}} + \sigma^2_{p_{z0_t}}) \\
&\quad + \frac{c^4}{E_0^4} (24\sigma^4_{p_{i\perp}} + 8\sigma^2_{p_{i\perp}} \sigma^2_{p_{z0_t}} + 8\sigma^2_{p_{i\perp}} \sigma^2_{z0_t} + 4\sigma^2_{p_{i\perp}} \sigma^2_{z0_t} + 3\sigma^2_{p_{z0_t}}) \right),
\end{align*}
\]
The rms $x$ and $y$ emittances are, keeping terms only up to order $1/E_{0t}^6$,

$$
\langle x(t)p_x(t) \rangle = \langle y(t)p_y(t) \rangle \approx \frac{c^2 \sigma_{p_{\perp}}^2 t}{E_{0t}} \left( 1 - \frac{c^2}{2E_{0t}^2} (4\sigma_{p_{\perp}}^2 + \sigma_{p_{\parallel}}^2) \right.
+ \frac{3c^4}{8E_{0t}^4} \left( 24\sigma_{p_{\perp}}^4 + 8\sigma_{p_{\perp}}^2 \sigma_{p_{\parallel}}^2 + 8\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 4\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 3\sigma_{p_{\parallel}}^4 \right),
$$

$$
\langle x(t)p_x(t) \rangle^2 = \langle y(t)p_y(t) \rangle^2 \approx \frac{c^4 \sigma_{p_{\perp}}^4 t^2}{E_{0t}^2} \left( 1 - \frac{c^2}{2E_{0t}^2} (4\sigma_{p_{\perp}}^2 + \sigma_{p_{\parallel}}^2) \right.
+ \frac{c^4}{E_{0t}^2} \left( 20\sigma_{p_{\perp}}^4 + 8\sigma_{p_{\perp}}^2 \sigma_{p_{\parallel}}^2 + 6\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 3\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 5\sigma_{p_{\parallel}}^4/2 \right). 
$$

The emittance grows quadratically with time with a coefficient that is of fourth order of smallness. There is no growth of the $x$ or $y$ emittance at second order of smallness, which order corresponds to the first-order term in the expansion of the square root in $1/E \propto 1/\sqrt{1+\Delta} \approx 1 - \Delta/2$. This has led to the statement that there is no emittance growth in “linear” beam transport although I find this use of the adjective “linear” to be obscure.

We now turn to longitudinal quantities:

$$
\langle p_z(t) \rangle = p_{0t,z}, \quad \langle p_z^2(t) \rangle = p_{0t,z}^2 + \sigma_{p_{\parallel}}^2;
$$

$$
\langle \Delta p_z^2(t) \rangle \equiv \langle (p_z(t) - \langle p_z(t) \rangle)^2 \rangle = \langle p_z^2(t) \rangle - \langle p_z(t) \rangle^2 = \sigma_{p_{\parallel}}^2;
$$

$$
\langle z(t) \rangle \approx \frac{c^2 t p_{0t,z}}{E_{0t}} \left( 1 - \frac{c^2}{2E_{0t}^2} (2\sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2) \right.
+ \frac{3c^4}{8E_{0t}^4} \left( 8\sigma_{p_{\parallel}}^4 + 12\sigma_{p_{\parallel}}^2 \sigma_{p_{\parallel}}^2 + 4\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 15\sigma_{p_{\parallel}}^4 \right),
$$

$$
\langle z(t)^2 \rangle \approx \frac{c^4 t^2 p_{0t,z}}{E_{0t}^2} \left( 1 - \frac{c^2}{E_{0t}^2} (2\sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2) \right.
+ \frac{c^4}{E_{0t}^2} \left( 7\sigma_{p_{\parallel}}^4 + 12\sigma_{p_{\parallel}}^2 \sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 27\sigma_{p_{\parallel}}^4/2 \right),
$$

$$
\langle z^2(t) \rangle \approx \sigma_{z_{\parallel}}^2 + \frac{c^4 t^2 p_{0t,z}}{E_{0t}^2} \left( 1 - \frac{c^2}{2E_{0t}^2} (2\sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2) \right.
+ \frac{3c^4}{8E_{0t}^4} \left( 8\sigma_{p_{\parallel}}^4 + 12\sigma_{p_{\parallel}}^2 \sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2 p_{0t,z}^2 + 27\sigma_{p_{\parallel}}^4/2 \right)
+ \frac{c^4 p_{\parallel}^2 t^2}{E_{0t}^2} \left( 1 - \frac{c^2}{2E_{0t}^2} (2\sigma_{p_{\parallel}}^2 + 3\sigma_{p_{\parallel}}^2 + 2p_{0t,z}^2) \right).
$$
\[
\langle \Delta z^2(t) \rangle \equiv \langle (z(t) - \langle z(t) \rangle)^2 \rangle = \langle z^2(t) \rangle - \langle z(t) \rangle^2 \\
\approx \sigma_z^2 + \frac{c^4 E_0^4}{E_0^6} \left( \frac{c^8 t^2 p_{0,z}^2 \sigma_{p_z}^4}{E_0^6} \sigma_{p_z}^4 \right) + \frac{c^4 \sigma_{p_z}^4 t^2}{E_0^4} \left( 1 - \frac{c^2}{E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{c^4}{E_0^4} \left( 8 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 8 \sigma_{p_z}^2 \sigma_{p_z}^2 + 5 \sigma_{p_z}^2 \sigma_{p_z}^2 + 2 + 15 \sigma_{p_z}^4 + p_{0,z}^4 \right) \right),
\]

\[
\langle \Delta z^2(t) \rangle \langle \Delta p_z^2(t) \rangle \approx \sigma_z^2 \sigma_{p_z}^2 + \frac{c^4 E_0^4}{E_0^6} \left( \frac{c^8 t^2 p_{0,z}^2 \sigma_{p_z}^4}{E_0^6} \sigma_{p_z}^4 \right) + \frac{c^4 \sigma_{p_z}^4 t^2}{E_0^4} \left( 1 - \frac{c^2}{E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{c^4}{E_0^4} \left( 8 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 8 \sigma_{p_z}^2 \sigma_{p_z}^2 + 5 \sigma_{p_z}^2 \sigma_{p_z}^2 + 2 + 15 \sigma_{p_z}^4 + p_{0,z}^4 \right) \right),
\]

\[
\langle z(t) \rangle \langle p_z(t) \rangle \approx \frac{c^2 t p_{0,z}^2}{E_0^2} \left( 1 - \frac{c^2}{2 E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{3 c^4}{8 E_0^4} \left( 8 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 8 \sigma_{p_z}^2 \sigma_{p_z}^2 + 24 \sigma_{p_z}^2 \sigma_{p_z}^2 + 15 \sigma_{p_z}^4 \right) \right),
\]

\[
\langle z(t) \rangle \langle p_z(t) \rangle \approx \frac{c^2 t p_{0,z}^2}{E_0^2} \left( 1 - \frac{c^2}{2 E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{3 c^4}{8 E_0^4} \left( 8 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 4 \sigma_{p_z}^2 \sigma_{p_z}^2 + 4 \sigma_{p_z}^2 \sigma_{p_z}^2 + 15 \sigma_{p_z}^4 \right) \right),
\]

\[
\langle \Delta z(t) \Delta p_z(t) \rangle = \langle (z(t) - \langle z(t) \rangle)(p_z(t) - \langle p_z(t) \rangle) \rangle \\
= \langle z(t) \rangle \langle p_z(t) \rangle - \langle z(t) \rangle \langle p_z(t) \rangle \\
\approx \frac{c^2 \sigma_{p_z}^4 t}{E_0^4} \left( 1 - \frac{c^2}{2 E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{3 c^4}{8 E_0^4} \left( 8 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 8 \sigma_{p_z}^2 \sigma_{p_z}^2 + 20 \sigma_{p_z}^2 \sigma_{p_z}^2 + 15 \sigma_{p_z}^4 \right) \right),
\]

\[
\langle \Delta z(t) \Delta p_z(t) \rangle \approx \frac{c^4 \sigma_{p_z}^4 t^2}{E_0^2} \left( 1 - \frac{c^2}{2 E_0^2} \left( 2 \sigma_{p_z}^2 + 3 \sigma_{p_z}^2 + 2 p_{0,z}^2 \right) \right) \\
+ \frac{c^4}{E_0^4} \left( 7 \sigma_{p_z}^2 + 12 \sigma_{p_z}^2 \sigma_{p_z}^2 + 8 \sigma_{p_z}^2 \sigma_{p_z}^2 + 18 \sigma_{p_z}^2 \sigma_{p_z}^2 + 27 \sigma_{p_z}^4 / 2 + p_{0,z}^4 \right) \right). \tag{7}
\]

The rms z emittance is, keeping terms only up to order \(1/E_0^6\),

\[
\epsilon_z(t) = \sqrt{\langle \Delta z^2(t) \rangle \langle \Delta p_z^2(t) \rangle - \langle \Delta z(t) \Delta p_z(t) \rangle}\]
symmetry, the 6 variable. For the present case of a field-free drift with no initial cross correlations and $x$ where $x=0$

$$\sigma \approx \sigma_{zt} \sigma_{pz} \left[ 1 + \frac{c^2 t^2}{2 E_0^2 \sigma_{zt}^2 \sigma_{pz}^2} \left( \sigma_{p_{zt}}^2 \sigma_{pz}^2 (p_{0t}^2 + \sigma_{pz}^2) + \sigma_{pz}^6 \frac{15 p_{0t}^2 + 3 \sigma_{pz}^2}{2} \right) \right]. \quad (8)$$

If $\sigma_{pz}^2 \ll p_{0t}^2$ the forms of the emittances (6) and (8) can be simplified accordingly.

### 2.1.1 Eigenemittances

It has been suggested that it will somehow be advantageous to compute the so-called eigenemittances of the beam transport. See, for example, sec. 26.3 of [6].

We consider the $6 \times 6$ second-moment matrix $\Sigma$ defined by,

$$\Sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = \langle \Delta x_i \Delta x_j \rangle,$$  \quad (9)

where $x_1 = x$, $x_2 = p_x$, $x_3 = y$, $x_4 = p_y$, $x_5 = z$, $x_6 = p_z$ when using $t$ as the independent variable. For the present case of a field-free drift with no initial cross correlations and $x$-$y$ symmetry, the $6 \times 6$ matrix $\Sigma$ is block diagonal with three $2 \times 2$ submatrices

$$\Sigma(t) = \begin{pmatrix} \Sigma_x(t) & 0 & 0 \\ 0 & \Sigma_y(t) & 0 \\ 0 & 0 & \Sigma_z(t) \end{pmatrix}, \quad (10)$$

$$\Sigma_x(t) = \Sigma_y(t) = \begin{pmatrix} \langle x^2(t) \rangle & \langle x(t)p_x(t) \rangle \\ \langle x(t)p_x(t) \rangle & \sigma_{p_{zt}}^2 \end{pmatrix}, \quad (11)$$

$$\Sigma_z(t) = \begin{pmatrix} \langle \Delta z^2(t) \rangle & \langle \Delta z(t) \Delta p_z(t) \rangle \\ \langle \Delta z(t) \Delta p_z(t) \rangle & \sigma_{pz}^2 - p_{0z}^2 \end{pmatrix}. \quad (12)$$

The eigenemittances at time $t$ are $|\lambda_i|$ where $\lambda_i$ are the eigenvalues of the matrix $J \Sigma(t)$, and,

$$J = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (13)$$

In the present case the matrix $J \Sigma(t)$ is block diagonal with the three $2 \times 2$ submatrices,

$$J_2 \Sigma_x(t) = J_2 \Sigma_y(t) = \begin{pmatrix} \langle x(t)p_x(t) \rangle & \sigma_{p_{zt}}^2 \\ -\langle x^2(t) \rangle & -\langle x(t)p_x(t) \rangle \end{pmatrix}, \quad (14)$$

$$J_2 \Sigma_z(t) = \begin{pmatrix} \langle \Delta z(t) \Delta p_z(t) \rangle & \sigma_{pz}^2 - p_{0z}^2 \\ -\langle \Delta z^2(t) \rangle & -\langle \Delta z(t) \Delta p_z(t) \rangle \end{pmatrix}. \quad (15)$$
whose eigenvalues are,

\[
\lambda_1 = -\lambda_2 = \lambda_3 = -\lambda_4 = i\sqrt{\langle x^2(t) \rangle \sigma_{\perp}^2 - \langle x(t)p_x(t) \rangle^2} = i\epsilon_x = i\epsilon_y, \tag{16}
\]
\[
\lambda_5 = -\lambda_6 = i\sqrt{\langle \Delta z^2(t) \rangle (\sigma_z^2 - \bar{p}_{0t,z}^2) - \langle \Delta z(t)\Delta p_z(t) \rangle^2} = i\epsilon_z. \tag{17}
\]

It appears to me that eigenemittances are the same as rms emittances in the present example.

References


