Emittance Growth
from Weak Relativistic Effects

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1 Problem

Although phase volume is invariant under canonical transformations of a Hamiltonian system (see, for example, [1]), approximations to phase volume such as rms emittance are not. Deduce approximate expressions for the growth of the various 2-D rms emittances of a Gaussian bunch of particles of mass m and charge q, initially centered on the origin and with $\langle p_x \rangle = 0 = \langle p_y \rangle$ but with nonzero energy spread about a central energy E_0, as this bunch propagates in a region of zero electromagnetic field. For the longitudinal emittance, consider both coordinates (z, p_z) and $(t, p_t = -E_{\rm total})$ [1].

2 Solution

This solution is an extension of [2, 3, 4]. Numerical examples of rms emittance growth during propagation of a "beam" in a field-free region are given on slide 8 of [5].

2.1 Emittances when t Is the Independent Variable

When using time t as the independent variable the canonical coordinates are $x, y, z, p_x, p_y, p_z)$ and the initial conditions are at time $t = 0 \equiv 0_t$. We suppose the initial bunch is Gaussian, with nonzero first and second moments,

$$\langle p_z(t=0)\rangle \equiv \langle p_z(0_t)\rangle = p_{0_t,z}\,, \qquad E_{0_t} \equiv c\sqrt{m^2c^2 + p_{0_t,z}^2} < \langle E(0_t)\rangle\,,$$
$$\left\langle x^2(0_t)\right\rangle = \left\langle y^2(0_t)\right\rangle = \sigma_{\perp_t}^2\,, \qquad \left\langle z^2(0_t)\right\rangle = \sigma_{z_t}^2\,, \qquad \left\langle p_x^2(0_t)\right\rangle = \left\langle p_y^2(0_t)\right\rangle = \sigma_{p_{\perp_t}}^2\,,$$
$$\left\langle p_z^2(0_t) - p_{0_t,z}^2\right\rangle = \left\langle (p_z(0_t) - p_{0_t,z})^2\right\rangle = \sigma_{p_{z_t}}^2\,, \qquad \left\langle p_z^2(0_t)\right\rangle = p_{0_t,z}^2 + \sigma_{p_{z_t}}^2\,. \tag{1}$$

That is, there are no cross correlations initially. We will also need to know some higher moments, such as,

$$\left\langle p_x^4(0_t)\right\rangle = \left\langle p_y^4(0_t)\right\rangle = 3\sigma_{p_{\perp_t}}^4\,, \qquad \left\langle p_x^6(0_t)\right\rangle = \left\langle p_y^6(0_t)\right\rangle = 15\sigma_{p_{\perp_t}}^6\,,$$
$$\left\langle (p_z(0_t) - p_{0_t,z})^3\right\rangle = 0, \qquad \left\langle p_z^3(0_t)\right\rangle = p_{0_t,z}^3 + 3\sigma_{p_{z_t}}^2 p_{0_t,z}\,,$$
$$\left\langle p_z(0_t)(p_z^2(0_t) - p_{0_t,z}^2)\right\rangle = 3\sigma_{p_{z_t}}^2 p_{0_t,z}\,,$$
$$\left\langle (p_z(0_t) - p_{0_t,z})^4\right\rangle = 3\sigma_{p_{z_t}}^4\,, \qquad \left\langle p_z^4(0_t)\right\rangle = p_{0_t,z}^4 + 6\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 3\sigma_{p_{z_t}}^4\,,$$
$$\left\langle p_z^2(0_t)(p_z^2(0_t) - p_{0_t,z}^2)\right\rangle = 5\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 3\sigma_{p_{z_t}}^4\,, \qquad \left\langle (p_z^2(0_t) - p_{0_t,z}^2)^2\right\rangle = 4\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 3\sigma_{p_{z_t}}^4\,,$$
$$\left\langle (p_z(0_t) - p_{0_t,z})^5\right\rangle = 0, \qquad \left\langle p_z^5(0_t)\right\rangle = p_{0_t,z}^5 + 10\sigma_{p_{z_t}}^2 p_{0_t,z}^3 + 15\sigma_{p_{z_t}}^4 p_{0_t,z}\,,$$

$$
\begin{aligned}
\left\langle p_z(0_t)(p_z^2(0_t)-p_{0_t,z}^2)^2\right\rangle &= 4\sigma_{p_{z_t}}^2 p_{0_t,z}^3 + 15\sigma_{p_{z_t}}^4 p_{0_t,z}\,, \\
\left\langle (p_z(0_t)-p_{0_t,z})^6\right\rangle = 15\sigma_{p_{z_t}}^6\,, \qquad \left\langle p_z^6(0_t)\right\rangle &= p_{0_t,z}^6 + 15\sigma_{p_{z_t}}^2 p_{0_t,z}^4 + 45\sigma_{p_{z_t}}^4 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^6\,, \\
\left\langle p_z^2(0_t)(p_z^2(0_t)-p_{0_t,z}^2)^2\right\rangle &= 4\sigma_{p_{z_t}}^2 p_{0_t,z}^4 + 39\sigma_{p_{z_t}}^4 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^6\,, \quad (2)
\end{aligned}
$$

which follow from the assumption that $p_z(0_t)-p_{0_t,z}$ has a Gaussian distribution.

The equations of motion are,

$$
\begin{aligned}
p_x(t) &= p_x(0_t), \qquad p_y(t)=p_y(0_t), \qquad p_z(t)=p_z(0_t), \\
v_x(t) &= v_x(0_t), \qquad v_y(t)=v_y(0_t), \qquad v_z(t)=v_z(0_t), \\
E(t) &= E(0_t) = c\sqrt{m^2c^2+p_x^2(0_t)+p_y^2(0_t)+p_z^2(0_t)} \\
&= c\sqrt{m^2c^2+p_{0_t,z}^2+p_x^2(0_t)+p_y^2(0_t)+p_z^2(0_t)-p_{0_t,z}^2} \\
&= E_{0_t}\sqrt{1+\frac{c^2}{E_{0_t}^2}(p_x^2(0_t)+p_y^2(0_t)+p_z^2(0_t)-p_{0_t,z)}^2)}, \\
x_i(t) &= x_i(0_t)+v_i(t)t = x_i(0_t)+\frac{c^2tp_i(0_t)}{E(0_t)}\,, \\
x_i^2(t) &= x_i^2(0_t)+\frac{2c^2tx_i(0_t)p_i(0_t)}{E(0_t)}+\frac{c^4t^2p_i^2(0_t)}{E^2(0_t)}\,. \qquad (3)
\end{aligned}
$$

It turns out that we need to expand $1/E(0_t)$ to order $1/E_{0_t}^5$ (and $1/E^2(0_t)$ to order $1/E_{0_t}^6$),

$$
\begin{aligned}
\frac{1}{E(0_t)} &\approx \frac{1}{E_{0_t}}\Big\{1-\frac{c^2}{2E_{0_t}^2}(p_x^2(0_t)+p_y^2(0_t)+p_z^2(0_t)-p_{0_t,z}^2) \\
&\quad +\frac{3c^4}{8E_{0_t}^4}\left[p_x^4(0_t)+p_y^4(0_t)+(p_z^2(0_t)-p_{0_t,z}^2)^2\right. \\
&\quad \left. +2p_x^2(0_t)p_y^2(0_t)+2(p_x^2(0_t)+p_y^2(0_t))(p_z^2(0_t)-p_{0_t,z}^2)\right]\Big\}, \\
\frac{1}{E^2(0_t)} &\approx \frac{1}{E_{0_t}^2}\Big\{1-\frac{c^2}{E_{0_t}^2}(p_x^2(0_t)+p_y^2(0_t)+p_z^2(0_t)-p_{0_t,z}^2) \\
&\quad +\frac{c^4}{E_{0_t}^4}\left[p_x^4(0_t)+p_y^4(0_t)+(p_z^2(0_t)-p_{0_t,z}^2)^2\right. \\
&\quad \left. +2p_x^2(0_t)p_y^2(0_t)+2(p_x^2(0_t)+p_y^2(0_t))(p_z^2(0_t)-p_{0_t,z}^2)\right]\Big\}. \qquad (4)
\end{aligned}
$$

Note that we expand in the small quantity $p_z^2(0_t)-p_{0_t,z}^2$ rather than in $\Delta p_z = p_z(0_t)-p_{0_t,z}$.

The nonzero first and second moments in x and y at time t are,

$$
\begin{aligned}
\left\langle p_x^2(t)\right\rangle &= \left\langle p_y^2(t)\right\rangle = \sigma_{p_{\perp_t}}^2, \\
\left\langle x^2(t)\right\rangle &= \left\langle y^2(t)\right\rangle \approx \sigma_{\perp_t}^2 + \frac{c^4\sigma_{p_{\perp_t}}^2 t^2}{E_{0_t}^2}\Bigg(1-\frac{c^2}{E_{0_t}^2}(4\sigma_{p_{\perp_t}}^2+\sigma_{p_{z_t}}^2) \\
&\quad +\frac{c^4}{E_{0_t}^4}(24\sigma_{p_{\perp_t}}^4+8\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2+8\sigma_{p_{\perp_t}}^2p_{0_t,z}^2+4\sigma_{p_{z_t}}^2p_{0_t,z}^2+3\sigma_{p_{z_t}}^4)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\langle x(t)p_x(t)\rangle &= \langle y(t)p_y(t)\rangle \approx \frac{c^2\sigma^2_{p_{\perp_t}} t}{E_{0_t}}\left(1 - \frac{c^2}{2E^2_{0_t}}(4\sigma^2_{p_{\perp_t}} + \sigma^2_{p_{z_t}})\right. \\
&\quad \left. + \frac{3c^4}{8E^4_{0_t}}(24\sigma^4_{p_{\perp_t}} + 8\sigma^2_{p_{\perp_t}}\sigma^2_{p_{z_t}} + 8\sigma^2_{p_{\perp_t}}p^2_{0_t,z} + 4\sigma^2_{p_{z_t}}p^2_{0_t,z} + 3\sigma^4_{p_{z_t}})\right), \\
\langle x(t)p_x(t)\rangle^2 &= \langle y(t)p_y(t)\rangle^2 \approx \frac{c^4\sigma^4_{p_{\perp_t}} t^2}{E^2_{0_t}}\left(1 - \frac{c^2}{E^2_{0_t}}(4\sigma^2_{p_{\perp_t}} + \sigma^2_{p_{z_t}})\right. \\
&\quad \left. + \frac{c^4}{E^4_{0_t}}(20\sigma^4_{p_{\perp_t}} + 8\sigma^2_{p_{\perp_t}}\sigma^2_{p_{z_t}} + 6\sigma^2_{p_{\perp_t}}p^2_{0_t,z} + 3\sigma^2_{p_{z_t}}p^2_{0_t,z} + 5\sigma^4_{p_{z_t}}/2)\right). \qquad (5)
\end{aligned}
$$

The rms x and y emittances are, keeping terms only up to order $1/E^6_{0_t}$,

$$
\begin{aligned}
\epsilon_x(t) &= \epsilon_y(t) = \sqrt{\langle x^2(t)\rangle \langle p^2(t)_x\rangle - \langle x(t)p_x(t)\rangle^2} \\
&\approx \sqrt{\sigma^2_{\perp_t}\sigma^2_{p_{\perp_t}} + \frac{c^8\sigma^4_{p_{\perp_t}} t^2}{E^6_{0_t}}(4\sigma^4_{p_{\perp_t}} + 2\sigma^2_{p_{\perp_t}}p^2_{0_t,z} + \sigma^2_{p_{z_t}}p^2_{0_t,z} + \sigma^4_{p_{z_t}}/2)} \\
&\approx \sigma_{\perp_t}\sigma_{p_{\perp_t}}\left[1 + \frac{c^8\sigma^2_{p_{\perp_t}} t^2}{2\sigma^2_{\perp_t}E^6_{0_t}}(4\sigma^4_{p_{\perp_t}} + 2\sigma^2_{p_{\perp_t}}p^2_{0_t,z} + \sigma^2_{p_{z_t}}p^2_{0_t,z} + \sigma^4_{p_{z_t}}/2)\right]. \qquad (6)
\end{aligned}
$$

The emittance grows quadratically with time with a coefficient that is of fourth order of smallness. There is no growth of the x or y emittance at second order of smallness, which order corresponds to the first-order term in the expansion of the square root in $1/E \propto 1/\sqrt{1+\Delta} \approx 1 - \Delta/2$. This has led to the statement that there is no emittance growth in "linear" beam transport *although I find this use of the adjective "linear" to be obscure..*

We now turn to longitudinal quantities:

$$
\begin{aligned}
\langle p_z(t)\rangle &= p_{0_t,z}, \qquad \left\langle p^2_z(t)\right\rangle = p^2_{0_t,z} + \sigma^2_{p_{z_t}}, \\
\left\langle \Delta p^2_z(t)\right\rangle &\equiv \left\langle (p_z(t) - \langle p_z(t)\rangle)^2\right\rangle = \left\langle p^2_z(t)\right\rangle - \langle p_z(t)\rangle^2 = \sigma^2_{p_{z_t}}, \\
\langle z(t)\rangle &\approx \frac{c^2 t p_{0_t,z}}{E_{0_t}}\left(1 - \frac{c^2}{2E^2_{0_t}}(2\sigma^2_{p_{\perp_t}} + 3\sigma^2_{p_{z_t}})\right. \\
&\quad \left. + \frac{3c^4}{8E^4_{0_t}}(8\sigma^4_{p_{\perp_t}} + 12\sigma^2_{p_{\perp_t}}\sigma^2_{p_{z_t}} + 4\sigma^2_{p_{z_t}}p^2_{0_t,z} + 15\sigma^4_{p_{z_t}})\right), \\
\langle z(t)\rangle^2 &\approx \frac{c^4 t^2 p^2_{0_t,z}}{E^2_{0_t}}\left(1 - \frac{c^2}{E^2_{0_t}}(2\sigma^2_{p_{\perp_t}} + 3\sigma^2_{p_{z_t}})\right. \\
&\quad \left. + \frac{c^4}{E^4_{0_t}}(7\sigma^4_{p_{\perp_t}} + 12\sigma^2_{p_{\perp_t}}\sigma^2_{p_{z_t}} + 3\sigma^2_{p_{z_t}}p^2_{0_t,z} + 27\sigma^4_{p_{z_t}}/2)\right), \\
\left\langle z^2(t)\right\rangle &\approx \sigma^2_{z_t} + \frac{c^4 t^2 p^2_{0_t,z}}{E^2_{0_t}}\left(1 - \frac{c^2}{E^2_{0_t}}(2\sigma^2_{p_{\perp_t}} + 3\sigma^2_{p_{z_t}})\right. \\
&\quad \left. + \frac{c^4}{E^4_{0_t}}(8\sigma^4_{p_{\perp_t}} + 12\sigma^2_{p_{\perp_t}}\sigma^2_{p_{z_t}} + 3\sigma^2_{p_{z_t}}p^2_{0_t,z} + 27\sigma^4_{p_{z_t}}/2\right) \\
&\quad + \frac{c^4\sigma^2_{p_{z_t}} t^2}{E^2_{0_t}}\left(1 - \frac{c^2}{E^2_{0_t}}(2\sigma^2_{p_{\perp_t}} + 3\sigma^2_{p_{z_t}} + 2p^2_{0_t,z})\right.
\end{aligned}
$$

$$
\begin{aligned}
& \quad + \frac{c^4}{E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 51\sigma_{p_{z_t}}^2 p_{0_t,z}^2/2 + 15\sigma_{p_{z_t}}^4 + p_{0_t,z}^4)\Bigg), \\
\left\langle \Delta z^2(t) \right\rangle &\equiv \left\langle (z(t) - \langle z(t) \rangle)^2 \right\rangle = \left\langle z^2(t) \right\rangle - \langle z(t) \rangle^2 \\
&\approx \sigma_{z_t}^2 + \frac{c^8 t^2 p_{0_t,z}^2 \sigma_{p_{\perp_t}}^4}{E_{0_t}^6} + \frac{c^4 \sigma_{p_{z_t}}^2 t^2}{E_{0_t}^2}\Bigg(1 - \frac{c^2}{E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2 + 2p_{0_t,z}^2) \\
& \quad + \frac{c^4}{E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 51\sigma_{p_{z_t}}^2 p_{0_t,z}^2/2 + 15\sigma_{p_{z_t}}^4 + p_{0_t,z}^4)\Bigg), \\
\left\langle \Delta z^2(t) \right\rangle \left\langle \Delta p_z^2(t) \right\rangle &\approx \sigma_{z_t}^2\sigma_{p_{z_t}}^2 + \frac{c^8 t^2 p_{0_t,z}^2 \sigma_{p_{z_t}}^2 \sigma_{p_{\perp_t}}^4}{E_{0_t}^6} + \frac{c^4 \sigma_{p_{z_t}}^4 t^2}{E_{0_t}^2}\Bigg(1 - \frac{c^2}{E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2 + 2p_{0_t,z}^2) \\
& \quad + \frac{c^4}{E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 51\sigma_{p_{z_t}}^2 p_{0_t,z}^2/2 + 15\sigma_{p_{z_t}}^4 + p_{0_t,z}^4)\Bigg), \\
\langle z(t) p_z(t) \rangle &\approx \frac{c^2 t p_{0_t,z}^2}{E_{0_t}}\Bigg(1 - \frac{c^2}{2E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2) \\
& \quad + \frac{3c^4}{8E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 24\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^4)\Bigg), \\
& \quad + \frac{c^2 \sigma_{p_{z_t}}^2 t}{E_{0_t}}\Bigg(1 - \frac{c^2}{2E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2 + 2p_{0_t,z}^2) \\
& \quad + \frac{3c^4}{8E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 4\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^4)\Bigg), \\
\langle z(t) \rangle \langle p_z(t) \rangle &\approx \frac{c^2 t p_{0_t,z}^2}{E_{0_t}}\Bigg(1 - \frac{c^2}{2E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2) \\
& \quad + \frac{3c^4}{8E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 4\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^4)\Bigg), \\
\langle \Delta z(t) \Delta p_z(t) \rangle &= \langle (z(t) - \langle z(t) \rangle)(p_z(t) - \langle p_z(t) \rangle) \rangle \\
&= \langle z(t) p_z(t) \rangle - \langle z(t) \rangle \langle p_z(t) \rangle \\
&\approx \frac{c^2 \sigma_{p_{z_t}}^2 t}{E_{0_t}}\Bigg(1 - \frac{c^2}{2E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2 + 2p_{0_t,z}^2) \\
& \quad + \frac{3c^4}{8E_{0_t}^4}(8\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 20\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 15\sigma_{p_{z_t}}^4)\Bigg), \\
\langle \Delta z(t) \Delta p_z(t) \rangle^2 &\approx \frac{c^4 \sigma_{p_{z_t}}^4 t^2}{E_{0_t}^2}\Bigg(1 - \frac{c^2}{E_{0_t}^2}(2\sigma_{p_{\perp_t}}^2 + 3\sigma_{p_{z_t}}^2 + 2p_{0_t,z}^2) \qquad (7) \\
& \quad + \frac{c^4}{E_{0_t}^4}(7\sigma_{p_{\perp_t}}^4 + 12\sigma_{p_{\perp_t}}^2\sigma_{p_{z_t}}^2 + 8\sigma_{p_{\perp_t}}^2 p_{0_t,z}^2 + 18\sigma_{p_{z_t}}^2 p_{0_t,z}^2 + 27\sigma_{p_{z_t}}^4/2 + p_{0_t,z}^4)\Bigg).
\end{aligned}
$$

The rms z emittance is, keeping terms only up to order $1/E_{0_t}^6$,

$$
\epsilon_z(t) = \sqrt{\langle \Delta z^2(t) \rangle \langle \Delta p_z^2(t) \rangle - \langle \Delta z(t) \Delta p_z(t) \rangle^2}
$$

$$\approx \sqrt{\sigma_{z_t}^2\sigma_{p_{z_t}}^2 + \frac{c^8t^2}{E_{0_t}^6}\left(\sigma_{p_{\perp t}}^4\sigma_{p_{z_t}}^2(p_{0_t,z}^2+\sigma_{p_{z_t}}^2) + \sigma_{p_{z_t}}^6\frac{15p_{0_t,z}^2+3\sigma_{p_{z_t}}^2}{2}\right)}$$

$$\approx \sigma_{z_t}\sigma_{p_{z_t}}\left[1 + \frac{c^8t^2}{2E_{0_t}^6\sigma_{z_t}^2\sigma_{p_{z_t}}^2}\left(\sigma_{p_{\perp t}}^4\sigma_{p_{z_t}}^2(p_{0_t,z}^2+\sigma_{p_{z_t}}^2) + \sigma_{p_{z_t}}^6\frac{15p_{0_t,z}^2+3\sigma_{p_{z_t}}^2}{2}\right)\right]. \quad (8)$$

If $\sigma_{p_{z_t}}^2 \ll p_{0_t,z}^2$ the forms of the emittances (6) and (8) can be simplified accordingly.

2.1.1 Eigenemittances

It has been suggested that it will somehow be advantageous to compute the so-called eigen-emittances of the beam transport. See, for example, sec. 26.3 of [6].

We consider the 6×6 second-moment matrix Σ defined by,

$$\Sigma_{ij} = \langle x_i x_j\rangle - \langle x_i\rangle\langle x_j\rangle = \langle \Delta x_i \Delta x_j\rangle, \quad (9)$$

where $x_1 = x$, $x_2 = p_x$, $x_3 = y$, $x_4 = p_y$, $x_5 = z$, $x_6 = p_z$ when using t as the independent variable. For the present case of a field-free drift with no initial cross correlations and x-y symmetry, the 6×6 matrix Σ is block diagonal with three 2×2 submatrices

$$\Sigma(t) = \begin{pmatrix} \Sigma_x(t) & 0 & 0 \\ 0 & \Sigma_y(t) & 0 \\ 0 & 0 & \Sigma_z(t) \end{pmatrix}, \quad (10)$$

$$\Sigma_x(t) = \Sigma_y(t) = \begin{pmatrix} \langle x^2(t)\rangle & \langle x(t)p_x(t)\rangle \\ \langle x(t)p_x(t)\rangle & \sigma_{p_\perp}^2 \end{pmatrix}, \quad (11)$$

$$\Sigma_z(t) = \begin{pmatrix} \langle \Delta z^2(t)\rangle & \langle \Delta z(t)\Delta p_z(t)\rangle \\ \langle \Delta z(t)\Delta p_z(t)\rangle & \sigma_{p_z}^2 - p_{0_t,z}^2 \end{pmatrix}. \quad (12)$$

The eigenemittances at time t are $|\lambda_i|$ where λ_i are the eigenvalues of the matrix $J\Sigma(t)$, and,

$$J = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (13)$$

In the present case the matrix $J\Sigma(t)$ is block diagonal with the three 2×2 submatrices,

$$J_2\Sigma_x(t) = J_2\Sigma_y(t) = \begin{pmatrix} \langle x(t)p_x(t)\rangle & \sigma_{p_\perp}^2 \\ -\langle x^2(t)\rangle & -\langle x(t)p_x(t)\rangle \end{pmatrix}, \quad (14)$$

$$J_2\Sigma_z(t) = \begin{pmatrix} \langle \Delta z(t)\Delta p_z(t)\rangle & \sigma_{p_z}^2 - p_{0_t,z}^2 \\ -\langle \Delta z^2(t)\rangle & -\langle \Delta z(t)\Delta p_z(t)\rangle \end{pmatrix}, \quad (15)$$

whose eigenvalues are,

$$\lambda_1 = -\lambda_2 = \lambda_3 = -\lambda_4 = i\sqrt{\langle x^2(t)\rangle\, \sigma_{p_\perp}^2 - \langle x(t) p_x(t)\rangle^2} = i\epsilon_x = i\epsilon_y, \tag{16}$$

$$\lambda_5 = -\lambda_6 = i\sqrt{\langle \Delta z^2(t)\rangle\, (\sigma_{p_z}^2 - p_{0_{t,z}}^2) - \langle \Delta z(t) \Delta p_z(t)\rangle^2} = i\epsilon_z. \tag{17}$$

It appears to me that eigenemittances are the same as rms emittances in the present example.

References

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