Group Velocity

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(December 4, 1996)

It seems that the concept of group velocity was first enunciated by Hamilton in 1841 in published abstracts of works that never appeared.[1] Hamilton considered a wave $\cos(kx - \omega t)$ defined only for negative x at $t = 0$ and incident on a dispersive medium that occupies the region $x > 0$. He concluded "that the velocity with which such vibration spreads into those portions of the vibratory medium which were previously undisturbed is in general different for the velocity of passage of a given phase from one particle to another within that portion of the medium which is already fully agitated; since we have velocity of transmission of phase $= \omega/k$ but velocity of propagation of vibrating motion $= d\omega/dk$." However, these results were largely ignored.

The group-velocity concept became widely known after being (re)introduced by Stokes in 1876 in a hydrodynamic context,[2] and the greater generality of the concept emphasized by Rayleigh in 1877 in sec. 191 of his book *The Theory of Sound*.[3] The early history of the group-velocity concept is well summarized in the book *The Propagation of Disturbances in Dispersive Media* by T.H. Havelock (Cambridge U. Press, 1914).[4]

I give two answers to the question of how one knows that wave energy propagates with the group velocity, both of which are "standard."[5] The discussion will be restricted to wave motion along the x axis for brevity.

1. The total energy E associated with a wave of amplitude $f(x,t)$ at a time t can in general be written as,

$$E(t) = \int (Af^2 + B\dot{f}^2)\, dx, \tag{1}$$

where $\dot{f} = \partial f/\partial t$ and either of A or B might be zero depending on the physical system. Typically, the term Af^2 is associated with energy stored in the wave medium due to the strain of the wave while $B\dot{f}^2$ is the kinetic energy of the medium due to the wave motion. For

[1] http://kirkmcd.princeton.edu/examples/optics/hamilton_pria_1_267_41.pdf
http://kirkmcd.princeton.edu/examples/optics/hamilton_pria_1_341_41.pdf
Hamiltion's longest discussion of group velocity was in a letter to Herschel (1839), p. 599 of
http://kirkmcd.princeton.edu/examples/mechanics/hamilton_papers_v2_583.pdf
Here he considered a wave with dispersion relation of the form $\omega = a \sin k$ for which the *rapidity with which a given phase travels* is $v_p = \omega/k$, while the velocity of *propagation of vibration* was found to be $a \cos k = d\omega/dk = v_{\rm group}$.

[2] Problem 11 of the Smith's Prize examination papers (Feb. 2, 1876), in *Mathematical and Physical Papers*, Vol. 5 (Johnson Reprint Co., New York, 1966), p. 362,
http://kirkmcd.princeton.edu/examples/fluids/stokes_mathematical_and_physical_papers_v5.pdf

[3] http://kirkmcd.princeton.edu/examples/mechanics/rayleigh_theory_of_sound_1.pdf
See also, http://kirkmcd.princeton.edu/examples/optics/rayleigh_nature_25_52_81.pdf

[4] http://kirkmcd.princeton.edu/examples/mechanics/havelock_waves_14.pdf

[5] A third generic argument has been given by G.B. Whitham, Comm. Pure Appl. Math. **14**, 675 (1961),
http://kirkmcd.princeton.edu/examples/fluids/whitham_cpam_14_675_61.pdf
See also sec. 2.1 of http://kirkmcd.princeton.edu/examples/biaxial.pdf

example, $B = 0$ for an electromagnetic wave while $A = 0$ for the granular systems recently studied by Swinney *et al.*, Nature **382**, 793 (1996).[6]

A question arises when one wishes to interpret the quantity $Af^2 + B\dot{f}^2$ as an energy density. As the wave changes with time it is possible that the energy moves in space. If the wave amplitude has the form of a traveling wave, $f(x - vt)$, then both f^2 and $\dot{f}^2$ are functions of a single variable, $x - vt$, and the energy can be said to be propagating with velocity v.

The concept of group velocity arises when a waveform is Fourier analyzed into a set of harmonic waves,

$$f(x,t) = \int F(k) e^{i(kx-\omega t)}\, dk, \tag{2}$$

characterized by wave number k and frequency $\omega(k)$ where the latter relation can be nontrivial due to dispersion in the wave medium. The harmonic wave of frequency ω has phase velocity $v_p = \omega/k$ which is not necessarily equal to the velocity v of the localized waveform. (In this discussion only the real part of f has physical significance.)

The spectral function $F(k)$ can be determined by the Fourier-inverse relation for the wave at a fixed time, say $t = 0$,

$$F(k) = \frac{1}{2\pi} \int f(x,0) e^{-ikx}\, dx. \tag{3}$$

However, we don't need to use this result in the present case.

The usual argument asks us to restrict our attention to waveforms whose spectral function $F(k)$ is narrow enough that the dispersion relation can be approximated as,

$$\omega = \omega(k_0) + \frac{d\omega(k_0)}{dk}(k - k_0), \tag{4}$$

i.e., the leading terms in a Taylor expansion about some central wave number k_0. (The sign of k_0 determines whether the pulse moves in the $+x$ or $-x$ direction.) Certainly this approximation breaks down for very short pulses in highly dispersive media.

In the approximation (4) we have,

$$\begin{aligned} f(x,t) &= e^{i[k_0(d\omega(k_0)/dk)-\omega_0]t} \int F(k) e^{ik[x-(d\omega(k_0)/dk)t]}\, dk \\ &= e^{i[k_0(d\omega(k_0)/dk)-\omega_0]t} f(x - (d\omega(k_0)/dk)t, 0). \end{aligned} \tag{5}$$

That is, to within a phase factor of unit modulus the waveform $f(x,t)$ is a function of a single variable, $x - (d\omega(k_0)/dk)t$, and so can be said to propagate with the group velocity,

$$v_{\rm group} = \frac{d\omega(k_0)}{dk}\,. \tag{6}$$

As argued above, the wave energy propagates with this velocity as well.

If the waveform is highly localized in space it will have a broad spectral content and the linear approximation to the dispersion relation may not suffice. If so, the waveform will change shape (disperse) as it propagates and the group velocity is not well defined.

[6] http://kirkmcd.princeton.edu/examples/fluids/umbanhowar_nature_382_793_96.pdf

This well-known argument appears to be due to Lord Kelvin, Proc. Roy. Soc. London **42**, 80 (1887),[7] and is reproduced in much the above form in sec. 3 of the book by Havelock.

2. In 1877 both Reynolds and Rayleigh published articles relating energy flow to group velocity. Reynolds' discussion[8] is based on water waves and can be found in sec. 273 of the book *Hydrodynamics* by H. Lamb, as well as in sec. 26 of *Mechanics of Deformable Bodies* by A. Sommerfeld (1946).

Rayleigh's argument[9] has been reprinted in the Appendix to Vol. 1 of his book *The Theory of Sound* and is based on the general observation that dispersion in a physical system is always accompanied by absorption. While the latter can often be ignored as a first approximation it should not be left out of discussions of energy flow.

Here, I repeat Rayleigh's argument, which seems to be less well-known than Kelvin's.

For a steady wave, energy is being transported into the medium at the same rate at which it is being absorbed, when averages are taken over a whole cycle of the wave. The power P absorbed by a mass m in the medium is $P = Fv$, where v is the velocity of the mass and $F = \gamma m v$ is the dissipative force. Thus, $P = \gamma m v^2$, and summing over all masses in some region, $P = 2\gamma T$, where T is the kinetic energy. Taking the time average, $\langle P \rangle = 2\gamma \langle T \rangle = \gamma E$, where E is the total energy. In writing $E = 2\langle T \rangle$ we suppose that the wave motion is a small departure from equilibrium so the restoring forces can be described by a quadratic potential, for which $E = 2\langle T \rangle$ according to the virial theorem of classical mechanics.

Consider a pure harmonic wave, $f = f_0 e^{i(kx-\omega t)}$, incident on a dispersive medium that occupies the half space $x > 0$. Because of absorption in the medium this wave dies out over some characteristic distance d. That is, the amplitude of the wave can be written $f_0 e^{-x/d}$ in the medium. Then the time average energy density is $\left\langle A f^2 + B\dot{f}^2 \right\rangle \equiv C f_0^2 e^{-2x/d}$.

The time-averaged power absorbed for $x > x_0$ in the medium is then,

$$\langle P(x_0) \rangle = \gamma \int_{x_0}^{\infty} \left\langle A f^2 + B\dot{f}^2 \right\rangle dx = C\gamma f_0^2 \int_{x_0}^{\infty} e^{-2x/d}\, dx = \frac{C\gamma d\, f_0^2}{2} e^{-2x_0/d}. \tag{7}$$

The (time-averaged) rate of energy flow per unit area across the plane $x = x_0$ is the (time-averaged) energy density there times the desired velocity of energy flow, v_E. The rate of energy flow is thus $C v_E f_0^2 e^{-2x_0/d}$. Comparing with eq. (7), we see that the energy flow velocity is given by,

$$v_E = \frac{\gamma d}{2}\,. \tag{8}$$

To find distance d, we suppose that in the absence of absorption the harmonic solutions obey a known dispersion relation, $k = k(\omega)$. Then the equation of motion including absorption, taken to be velocity dependent, differs from that without absorption only by replacing the second time derivative $\partial^2/\partial t^2$ with the form,

$$\frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial t}\,, \tag{9}$$

[7] http://kirkmcd.princeton.edu/examples/fluids/thomson_prsl_42_80_87.pdf

[8] http://kirkmcd.princeton.edu/examples/fluids/reynolds_nature_16_343_77.pdf

[9] http://kirkmcd.princeton.edu/examples/fluids/rayleigh_plms_9_21_77.pdf

where γ, whose dimensions are 1/time, characterizes the absorption process. The new dispersion relation that results on inserting our trial harmonic solution into the wave equation differs only in the term ω^2 being replaced by $\omega^2 + i\gamma\omega$. For weak absorption this is equivalent to replacing ω by $\omega + i\gamma/2$. The corresponding wave number is therefore $k(\omega + i\gamma/2) \approx k(\omega) + i(\gamma/2)(dk/d\omega)$, again ignoring terms of order γ^2. The wave solution in the presence of absorption is therefore approximately,

$$f(x > 0, t) = f_0 \, e^{-(\gamma/2)(dk/d\omega)x} e^{i(kx-\omega t)}. \tag{10}$$

Thus, the characteristic attenuation length is ,

$$d = \frac{2}{\gamma} \frac{d\omega}{dk} \, . \tag{11}$$

From eq. (8) the velocity of energy flow is,

$$v_E = \frac{d\omega}{dk} = v_{\rm group}. \tag{12}$$

An objection to this argument would be that it doesn't apply if the absorption is too strong. It may be that the heroic efforts of Sommerfeld and Brillouin (1914)[10] to clarify signal propagation in the case of highly absorptive anomalous dispersion in optical media (where $v_{\rm group}$ exceeds the speed of light) have left the impression that the more ordinary case is similarly intricate.

[10] L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, 1960), http://kirkmcd.princeton.edu/examples/EM/brillouin_waves_60.pdf