Gravitational Interaction between Fast-Moving Particles

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(June 24, 2018: updated September 13, 2018)

1 Problem

Discuss the gravitational interaction of a pair of electrically-neutral, fast-moving particles.

In the limit that the particles move with the speed c of light in vacuum, the discussion is applicable to photons and laser beams

2 Solution

The case of massless particles has been discussed in the literature [1]-[14] using general relativity. Here we show how a key result can be deduced via special relativity plus Newtonian gravitation, that the gravitational interaction is "zero" between particles that move in parallel at speeds close to c.

The (attractive) force between two particles of mass m at rest, separated by distance d is $F = Gm^2/d^2$, along their line of centers.

We now consider an observer the moves with speed $v \approx c$, with respect to the rest frame of the particles, in a direction perpendicular to their line of center, Of course, in the frame of the observer, the particles appear to have speed v.

The Minkowski 4-force between the particles in the $\star$ frame of the observer is,

$$f_\mu = (\gamma \mathbf{F}^\star \cdot \mathbf{v}/c, \gamma \mathbf{F}^\star), \tag{1}$$

where $\mathbf{v}$ is the velocity of the particles in the frame of the observer, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. In the rest frame of the particles, the 4-force is just $f_\mu = (0, \mathbf{F})$.

When the force $\mathbf{F}$ is perpendicular to $\mathbf{v}$, as considered above, the Lorentz transformation of the 4-force f_μ from the rest frame of the particles to the observer's frame tells us that,[1]

$$\gamma \mathbf{F}^\star = \mathbf{F}, \qquad \mathbf{F}^\star = \frac{\mathbf{F}}{\gamma} \qquad (\mathbf{F} \cdot \mathbf{v} = 0). \tag{2}$$

As $v \to c$ and γ grows large, the observer considers that the gravitational interaction between the particles (whose line of centers is perpendicular to $\mathbf{v}$) goes to zero.

We infer that for photons with $v = c$, moving parallel and "side by side", their gravitational interaction is zero.

This is equivalent to the result reported in [1] (and subsequent papers, such as [14]) that there is no gravitational interaction between parallel beams of light (moving in the same direction).

[1] Similarly, if $\mathbf{F}$ is parallel to $\mathbf{v}$, then $\gamma \mathbf{F}^\star = \gamma \mathbf{F}$, *i.e.*, $\mathbf{F}^\star = \mathbf{F}$.

2.1 Counterpropagating Particles

The case of particles moving at high speeds in opposite directions cannot be analyzed using special relativity only.

It is known from general relativity [16, 17] that a fast-moving particle that passes a particle at rest experiences a transverse "kick" (and an angular deflection) which is twice as large as expected from a Newtonian analysis.[2] So it is agreeable that when both particles have high speeds in opposite directions, the effect of each on the other is twice the Newtonian result, and the net angular deflection between the two particles is four times the Newtonian expectation.

2.2 Misner's Paradox

In 1959, Misner [18], citing Tolman [2], commented that if a positron from, say, a ^{22}Na β^+-source annihilated with an electron to produce two photons, there would be an increase of the gravitational effect on a distance observer. Likewise, when protons are accelerated to speeds near c in a synchrotron, the gravitational pull of the synchrotron on a distance observer would increase. Misner found such a time dependent gravitational effect "paradoxical", perhaps because it violates the equivalence principle, and sought to explain it away.

The view of this author is that the equivalence principle is only a low-velocity approximation, and that a distant observer could, in principle, detect changes in the gravitational effect of masses as their velocities change. Indeed, the detection of gravitational radiation by LIGO in black-hole mergers [19]-[25], in which the relative velocities of the black holes approach c just before the merger (and so the black holes are somewhat equivalent to photons with energies of many solar masses), is evidence that distant observers can detect velocity-dependent gravitational effects.

A Appendix: Electromagnetic Interactions of High-Speed Charged Particles

For comparison, we review the case of electrically charged particles with speeds close to c in the lab frame. See also [26].

The (Lorentz) force on an electric charge q with velocity $\mathbf{v}$ is, in Gaussian units,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{3}$$

where we suppose the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are due to another electric charge, say $-q$.

When the other charge has the same velocity $\mathbf{v}$ as the first, and is "side-by-side" with it, separated by distance d, we can consider the rest frame of the charges, where their separation is also d, and the fields of the other charge at the position of the first are,

$$\mathbf{E}^\star = \frac{q\,\hat{\mathbf{d}}}{d^2}, \qquad \mathbf{B}^\star = 0, \tag{4}$$

[2]For comments by the author on this famous result, see [15].

where unit vector $\hat{\mathbf{d}}$ points from the first charge to the second. Transforming the fields of the second charge at the position of the first to the lab frame, we have,

$$\mathbf{E} = \gamma \mathbf{E}^\star = \gamma \frac{q\,\hat{\mathbf{d}}}{d^2}\,, \qquad \mathbf{B} = \gamma \frac{\mathbf{v}}{c} \times \mathbf{E}^\star = \gamma \frac{q\mathbf{v} \times \hat{\mathbf{d}}}{d^2}\,. \tag{5}$$

Hence, the force on the first charge in the lab frame is

$$\mathbf{F} = \gamma q \left[\mathbf{E}^\star + \frac{\mathbf{v}}{c} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}^\star\right)\right] = \gamma q \mathbf{E}^\star \left(1 - \frac{v^2}{c^2}\right) = \frac{q\mathbf{E}^\star}{\gamma} = \frac{q^2\,\hat{\mathbf{d}}}{\gamma d^2}\,, \tag{6}$$

which goes to zero as $v \to c$. Thus, the electromagnetic interaction between side-by-side, fast-moving charged particles is negligible, and if photons carried electric charge, there would be no electromagnetic interaction between parallel beams of such photons.

References

[1] R.C. Tolman, P. Ehrenfest and B. Podolsky, *On the Gravitational Field Produced by Light*, Phys. Rev. **37**, 602 (1931), kirkmcd.princeton.edu/examples/GR/tolman_pr_37_602_31.pdf

[2] R.C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, 1934), p. 272 ff, http://kirkmcd.princeton.edu/examples/GR/tolman_rtc.pdf

[3] J.C. Hegarty, *Gravitational Effect of Electromagnetic Radiation*, Nuovo Cim. **61B**, 47 (1969), kirkmcd.princeton.edu/examples/GR/hegarty_nc_61b_47_69.pdf

[4] W.B. Bonnor, *The Gravitational Field of Light*, Comm. Math. Phys. **13**, 163 (1969), kirkmcd.princeton.edu/examples/GR/bonnor_cmp_13_163_69.pdf

[5] P.C. Aichelburg and R.U. Sexl, *On the gravitational field of a massless particle*, Gen. Rel. Grav. **2**, 303 (1971), kirkmcd.princeton.edu/examples/GR/aichelburg_grg_2_303_71.pdf

[6] N.A. Voronov and I.Y. Kobzarev, *On the gravitational field of a massless particle*, Sov. Phys. JETP **39**, 575 (1973), kirkmcd.princeton.edu/examples/GR/voronov_spjetp_39_575_73.pdf

[7] M.O. Scully, *General-relativistic treatment of the gravitational coupling between laser beams*, Phys. Rev. D **19**, 3582 (1979), kirkmcd.princeton.edu/examples/GR/scully_prd_19_3582_79.pdf

[8] T. Dray and G. 't Hooft, *The Gravitational Shock Wave of a Massless Particle*, Nucl. Phys. B **253**, 173 (1985), kirkmcd.princeton.edu/examples/GR/dray_np_b253_173_85.pdf

[9] U. von der Gönna and D. Kramer, *On the gravitational field of a massless particle*, Gen. Rel. Grav. **31**, 349 (1990), kirkmcd.princeton.edu/examples/GR/vandergonna_grg_31_349_90.pdf

[10] D. Kramer, *Two Counter-moving Light Beams*, Class. Quan. Grav. **15**, L73 (1998), kirkmcd.princeton.edu/examples/GR/kramer_cqg_15_L73_98.pdf

[11] B.V. Ivanov, *Colliding beams of light*, Class. Quan. Grav. **15**, L73 (1998),
kirkmcd.princeton.edu/examples/GR/ivanov_cqg_20_397_03.pdf

[12] W.B. Bonnor, *The gravitational field of photons*, Gen. Rel. Grav. **41**, 77 (2009),
kirkmcd.princeton.edu/examples/GR/bonnor_grg_41_77_09.pdf

[13] J.W. van Holten, *The gravitational field of a light wave*, Fort. Phys. **59**, 284 (2011),
kirkmcd.princeton.edu/examples/GR/vanholten_fp_59_284_11.pdf

[14] D. Rätzel, M. Wilkens and R. Menzel, *Gravitational properties of light–the gravitational field of a laser*, New J. Phys. **18**, 023009 (2016),
kirkmcd.princeton.edu/examples/GR/ratzel_njp_18_023009_16.pdf

[15] K.T. McDonald, *Gravitational Acceleration of a Moving Object at the Earth's Surface* (Apr. 4, 2016), kirkmcd.princeton.edu/examples/gravity_moving.pdf

[16] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, S.K.P. Akad. Wiss. 831 (1915),
kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15.pdf
Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity,
kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15_english.pdf

[17] A. Einstein, *Die Grundlage der allgemeinem Relativitätstheorie*, Ann. d. Phys. **49**, 769 (1916), kirkmcd.princeton.edu/examples/GR/einstein_ap_49_769_16.pdf
The Foundation of the General Theory of Relativity,
kirkmcd.princeton.edu/examples/GR/einstein_ap_49_769_16_english.pdf

[18] C. Misner and P. Putnam, *Active Gravitational Mass*, Phys. Rev. **116**, 1045 (1959),
kirkmcd.princeton.edu/examples/GR/misner_pr_116_1045_59.pdf

[19] B.P. Abbott *et al.*, *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116**, 061102 (2016),
kirkmcd.princeton.edu/examples/GR/abbott_prl_116_061102_16.pdf

[20] B.P. Abbott *et al.*, *GW150914: The Advanced LIGO Detectors in the Era of First Discoveries*, Phys. Rev. Lett. **116**, 131103 (2016),
kirkmcd.princeton.edu/examples/GR/abbott_prl_116_131103_16.pdf

[21] B.P. Abbott *et al.*, *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence*, Phys. Rev. Lett. **116**, 241103 (2016),
kirkmcd.princeton.edu/examples/GR/abbott_prl_116_241103_16.pdf

[22] B.P. Abbott *et al.*, *GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2*, Phys. Rev. Lett. **118**, 2221101 (2017),
kirkmcd.princeton.edu/examples/GR/abbott_prl_118_221101_17.pdf

[23] B.P. Abbott *et al.*, *GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence*, Phys. Rev. Lett. **119**, 141101 (2017),
kirkmcd.princeton.edu/examples/GR/abbott_prl_119_141101_17.pdf

[24] B.P. Abbott *et al.*, *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, Phys. Rev. Lett. **119**, 161101 (2017),
kirkmcd.princeton.edu/examples/GR/abbott_prl_119_161101_17.pdf

[25] B.P. Abbott *et al.*, *Gravitational Waves and Gamma-Rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A*, Ap. J. Lett. **848**, L13 (2017),
kirkmcd.princeton.edu/examples/GR/abbott_apjl_848_L13_17.pdf

[26] K.T. McDonald, *Force between Two Comoving Electric Charges* (May 24, 2014),
kirkmcd.princeton.edu/examples/comoving.pdf