The Golfer's Nemesis

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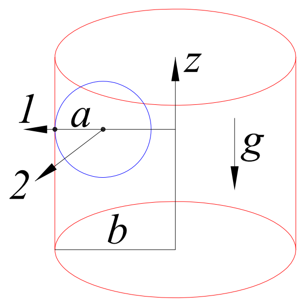
1 Problem

Can a golf ball roll into the cup, roll around on its vertical wall and pop back out?[1]

Consider a sphere of radius a that rolls without slipping inside a vertical cylinder of radius $b > a$.



If $\Omega = \dot{\phi}$ = angular velocity of the point of contact about the vertical, $\hat{\mathbf{1}}$ points from the center of the sphere to the point of contact, $\hat{\mathbf{z}}$ is vertical, and $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}$, show that the component equations of motion are,

$$\hat{\mathbf{z}}: \qquad \dot{\Omega} = 0, \tag{1}$$

$$\hat{\mathbf{1}}: \qquad a\,\dot{\omega}_1 = \Omega z, \tag{2}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot{z} = -ma^2 g - I a\,\omega_1\,\Omega. \tag{3}$$

Show that z of the center of mass executes simple harmonic motion, and if at $t = 0$, $z = 0$, $\dot{z} = \dot{z}_0$, and $\omega_1 = \omega_{10}$, then,

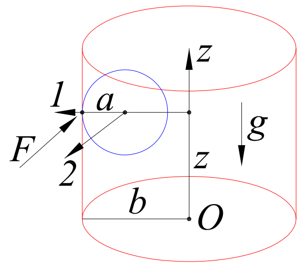
$$z = \frac{ma^2 g + I a\,\Omega\,\omega_{10}}{I\,\Omega^2}(\cos\alpha t - 1) + \frac{\dot{z}_0}{\alpha}\sin\alpha t, \qquad \text{where} \qquad \alpha = \Omega\sqrt{\frac{I}{I + ma^2}}. \tag{4}$$

With what velocity and angular velocity must the ball arrive at the rim of the cup to fall in and execute the above oscillatory motion, and possibly pop back out?

2 Solution

This problem is discussed in §421, p. 357 of E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948),
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf

[1]This behavior is distinct from the possibility that the ball bounces off the flagpole in the hole, or the plastic insert therein, as occurs from time to time.



We consider a sphere, of mass m and radius a with moment of inertia I about its center, that rolls without slipping on a fixed, vertical cylinder of radius $b > a$. We use a set of principal axes (but not body axes) about the center of the sphere of radius a, where $\hat{\mathbf{1}}$ points outward along the horizontal line from the center of the spheres to the point of contact with the cylinder. Axis $\hat{\mathbf{3}}$ is vertical (parallel to $\hat{\mathbf{z}}$), and axis $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}$ is also horizontal).

The center of the sphere of radius a is at position $\mathbf{r} = (b-a)\,\hat{\mathbf{1}} + z\,\hat{\mathbf{z}}$ with respect to the origin at the bottom center of the cylindere. Then, the velocity of the center of the sphere of radius a is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (b-a)\frac{d\hat{\mathbf{1}}}{dt} + \dot{z}\,\hat{\mathbf{z}}\,. \tag{5}$$

The (nonholonomic) constraint of rolling without slipping is that the point of contact of sphere with the cylinder is instantaneously at rest in the lab frame,

$$\mathbf{v}_{\rm contact} = 0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = (b-a)\frac{d\hat{\mathbf{1}}}{dt} + \dot{z}\,\hat{\mathbf{z}} + a\boldsymbol{\omega} \times \hat{\mathbf{1}}, \tag{6}$$

where $\boldsymbol{\omega}$ is the total angular velocity of the sphere in the lab frame, and $\mathbf{a} = a\,\hat{\mathbf{1}}$ is the vector from the center of the sphere of radius a to the point of contact.

The force and torque equations of motion for (the center of) the sphere of radius a are,

$$m\frac{d\mathbf{v}}{dt} = m(b-a)\frac{d^2\hat{\mathbf{1}}}{dt^2} + m\ddot{z}\,\hat{\mathbf{z}} = \mathbf{F} - mg\,\hat{\mathbf{z}}, \qquad \mathbf{F} = m(b-a)\frac{d^2\hat{\mathbf{1}}}{dt^2} + m(g+\ddot{z})\,\hat{\mathbf{z}}, \tag{7}$$

$$\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = \mathbf{a} \times \mathbf{F} = ma(b-a)\,\hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} - m(g+\ddot{z})a\,\hat{\mathbf{2}}, \tag{8}$$

where I is the moment of inertia of the sphere about its center.

We define $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{z}}$ as the angular velocity of the center of the sphere (and also of the point of contact, as well as of the triad $\hat{\mathbf{1}}$-$\hat{\mathbf{2}}$-$\hat{\mathbf{3}}$) about the vertical axis, such that,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\Omega} \times \mathbf{1} = \Omega\,\hat{\mathbf{2}}, \quad \frac{d^2\hat{\mathbf{1}}}{dt} = \dot{\Omega}\,\hat{\mathbf{2}} + \Omega\,\boldsymbol{\Omega} \times \hat{\mathbf{2}} = -\Omega^2\hat{\mathbf{1}} + \dot{\Omega}\hat{\mathbf{2}}, \quad \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt} = \dot{\Omega}\,\hat{\mathbf{z}}. \tag{9}$$

The velocity (5) of the center of the sphere can now be written as,

$$\mathbf{v} = -\Omega(b-a)\,\hat{\mathbf{2}} + \dot{z}\,\hat{\mathbf{z}}\,, \tag{10}$$

so the $\hat{\mathbf{2}}$-component of the total angular velocity $\boldsymbol{\omega}$ of the sphere about its center (and also that about the point of contact) is $v_z/a = \dot{z}/a$, and the $\hat{\mathbf{z}}$-component is $v_2/a = -(b-a)/a$. Thus,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \frac{\dot{z}}{a}\,\hat{\mathbf{2}} - \Omega\frac{b-a}{a}\,\hat{\mathbf{z}}, \qquad \frac{d\boldsymbol{\omega}}{dt} = \dot{\omega}_1\,\hat{\mathbf{1}} + \Omega\,\omega_1\,\hat{\mathbf{2}} + \frac{\ddot{z}}{a}\,\hat{\mathbf{2}} - \frac{\Omega\dot{z}}{a}\,\hat{\mathbf{1}} - \dot{\Omega}\frac{b-a}{a}\,\hat{\mathbf{z}}, \tag{11}$$

With these, the equation of motion (8) becomes,

$$I\left[\left(\dot\omega_1 - \frac{\Omega\dot z}{a}\right)\hat{\mathbf{1}} + \left(\Omega\,\omega_1 + \frac{\ddot z}{a}\right)\hat{\mathbf{2}} - \dot\Omega\frac{b-a}{a}\hat{\mathbf{z}}\right] = ma(b-a)\,\dot\Omega\,\hat{\mathbf{z}} - m(g+\ddot z)a\,\hat{\mathbf{2}}, \qquad (12)$$

The components of the equation of motion imply,

$$\hat{\mathbf{z}}: \qquad \dot\Omega = 0, \qquad \Omega = \text{constant}, \qquad (13)$$

$$\hat{\mathbf{1}}: \qquad \dot\omega_1 = \frac{\Omega\dot z}{a}, \qquad \omega_1 = \frac{\Omega z}{a} + \omega_{10}, \qquad (14)$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot z + I\,\Omega^2 z = -ma^2 g - I\,\Omega\,\omega_{10}. \qquad (15)$$

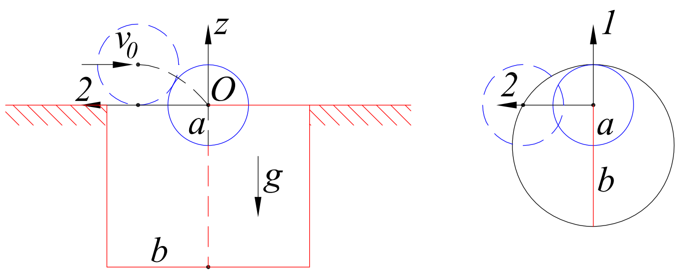
The center of the sphere executes simple harmonic motion in z,[2] and if at time $t = 0$, $z = 0$, $\dot z = \dot z_0$, $\omega_1 = \omega_{10}$, then,

$$z = \frac{ma^2 g + Ia\,\Omega\,\omega_{10}}{I\,\Omega^2}(\cos\alpha t - 1) + \frac{\dot z_0}{\alpha}\sin\alpha t, \qquad \text{where} \qquad \alpha = \Omega\sqrt{\frac{I}{I + ma^2}}\,. \qquad (16)$$

We now consider under what conditions a golf ball could roll into a cup/vertical cylinder such that at time $t = 0$ the motion is described by eq. (16).

According to eqs. (10) and (11), the velocity $\mathbf{v}_0$ and the angular velocity $\boldsymbol{\omega}_0$ at this time must be,

$$\mathbf{v}_0 = -\Omega(b-a)\,\hat{\mathbf{2}} + \dot z_0\,\hat{\mathbf{z}}\,, \qquad \boldsymbol{\omega}_0 = \omega_{10}\,\hat{\mathbf{1}} + \frac{\dot z_0}{a}\,\hat{\mathbf{2}} - \Omega\frac{b-a}{a}\,\hat{\mathbf{z}}. \qquad (17)$$



The figure above shows side and top views of the ball as it enters the cup, after rolling into it from the left while on the horizontal surface. At time $t = 0$, the ball has fallen through height a, so $\dot z_0 = -\sqrt{2ag}$. If the ball arrived at the top of the cup with horizontal velocity v_0 (in the $-\hat{\mathbf{2}}$ direction), then this is also the horizontal velocity when the center of the ball has fallen to $z = 0$, and so $\Omega = v_0/(b-a)$. The angular velocity of the ball did not change while it fell into the cup, so the angular velocity at the time of arrival was,

$$\boldsymbol{\omega}_{\text{arrival}} = \boldsymbol{\omega}_0 = \omega_{10}\,\hat{\mathbf{1}} - \sqrt{\frac{2g}{a}}\,\hat{\mathbf{2}} - \frac{v_0}{a}\,\hat{\mathbf{z}}, \qquad \mathbf{v}_{\text{arrival}} = -v_0\,\hat{\mathbf{2}} = -\Omega(b-a)\,\hat{\mathbf{2}}. \qquad (18)$$

If the ball had been simply rolling without slipping prior to arrival at the cup, then $\omega_{10} = v_0/a$ and the $\hat{\mathbf{2}}$- and $\hat{\mathbf{z}}$-components of $\boldsymbol{\omega}_{\text{arrival}}$ would be zero. Hence, only under special conditions

[2] This motion can be regarded as a nutation about steady motion with angular velocity Ω in a horizontal circle at $z = -(ma^2 g + Ia\,\Omega\,\omega_{10})/I\,\Omega^2$.

of rolling with slipping at the moment of arrival at the cup could the ball roll into it and pop back out after following motion of the form)16.

For a golf ball of uniform mass density, $I = 2ma^2/5$, and $\alpha = \sqrt{2/7}\,\Omega = \Omega/1.87$. If the golf ball does pop out of the hole, it does so in somewhat less than one period of the vertical oscillation, *i.e.*, in less the 1.87 revolutions of the ball around the vertical axis of the cup.