

# The Golfer's Nemesis

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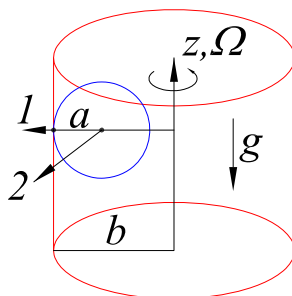
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## 1 Problem

Can a golf ball roll into the cup, roll around on its vertical wall and pop back out?<sup>1</sup>

Consider a sphere of radius  $a$  that rolls without slipping inside a vertical cylinder of radius  $b > a$ .



If  $\Omega = \dot{\phi}$  = angular velocity of the point of contact about the vertical,  $\hat{\mathbf{l}}$  points from the center of the sphere to the point of contact,  $\hat{\mathbf{z}}$  is vertical, and  $\hat{\mathbf{l}} = \hat{\mathbf{z}} \times \hat{\mathbf{l}}$ , show that the component equations of motion are,

$$\hat{\mathbf{z}} : \quad \dot{\Omega} = 0, \quad (1)$$

$$\hat{\mathbf{l}} : \quad a \dot{\omega}_1 = \Omega z, \quad (2)$$

$$\hat{\mathbf{l}} : \quad (I + ma^2) \ddot{z} = -ma^2g - Ia \omega_1 \Omega. \quad (3)$$

Show that  $z$  of the center of mass executes simple harmonic motion, and if at  $t = 0$ ,  $z = 0$ ,  $\dot{z} = \dot{z}_0$ , and  $\omega_1 = \omega_{10}$ , then,

$$z = \frac{ma^2g + Ia \Omega \omega_{10}}{I \Omega^2} (\cos \alpha t - 1) + \frac{\dot{z}_0}{\alpha} \sin \alpha t, \quad \text{where} \quad \alpha = \Omega \sqrt{\frac{I}{I + ma^2}}. \quad (4)$$

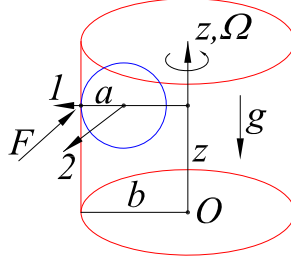
With what velocity and angular velocity must the ball arrive at the rim of the cup to fall in and execute the above oscillatory motion, and possibly pop back out?

## 2 Solution

This problem is discussed in §421, p. 357 of E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948),

[http://kirkmcd.princeton.edu/examples/mechanics/milne\\_mechanics.pdf](http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf)

<sup>1</sup>This behavior is distinct from the possibility that the ball bounces off the flagpole in the hole, or the plastic insert therein, as occurs from time to time.



We consider a sphere, of mass  $m$  and radius  $a$  with moment of inertia  $I$  about its center, that rolls without slipping on a fixed, vertical cylinder of radius  $b > a$ . We use a set of principal axes (but not body axes) about the center of the sphere of radius  $a$ , where  $\hat{\mathbf{1}}$  points outward along the horizontal line from the center of the spheres to the point of contact with the cylinder. Axis  $\hat{\mathbf{3}}$  is vertical (parallel to  $\hat{\mathbf{z}}$ ), and axis  $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}$  is also horizontal).

The center of the sphere of radius  $a$  is at position  $\mathbf{r} = (b - a) \hat{\mathbf{1}} + z \hat{\mathbf{z}}$  with respect to the origin at the bottom center of the cylinder. Then, the velocity of the center of the sphere of radius  $a$  is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (b - a) \frac{d\hat{\mathbf{1}}}{dt} + \dot{z} \hat{\mathbf{z}}. \quad (5)$$

The (nonholonomic) constraint of rolling without slipping is that the point of contact of sphere with the cylinder is instantaneously at rest in the lab frame,

$$\mathbf{v}_{\text{contact}} = 0 = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = (b - a) \frac{d\hat{\mathbf{1}}}{dt} + \dot{z} \hat{\mathbf{z}} + a \boldsymbol{\omega} \times \hat{\mathbf{1}}, \quad (6)$$

where  $\boldsymbol{\omega}$  is the total angular velocity of the sphere in the lab frame, and  $\mathbf{a} = a \hat{\mathbf{1}}$  is the vector from the center of the sphere of radius  $a$  to the point of contact.

The force and torque equations of motion for (the center of) the sphere of radius  $a$  are,

$$m \frac{d\mathbf{v}}{dt} = m(b - a) \frac{d^2\hat{\mathbf{1}}}{dt^2} + m\dot{z} \hat{\mathbf{z}} = \mathbf{F} - mg \hat{\mathbf{z}}, \quad \mathbf{F} = m(b - a) \frac{d^2\hat{\mathbf{1}}}{dt^2} + m(g + \ddot{z}) \hat{\mathbf{z}}, \quad (7)$$

$$\frac{d\mathbf{L}}{dt} = I \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = \mathbf{a} \times \mathbf{F} = ma(b - a) \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} - m(g + \ddot{z})a \hat{\mathbf{2}}, \quad (8)$$

where  $I$  is the moment of inertia of the sphere about its center.

We define  $\boldsymbol{\Omega} = \Omega \hat{\mathbf{z}}$  as the angular velocity of the center of the sphere (and also of the point of contact, as well as of the triad  $\hat{\mathbf{1}}\text{-}\hat{\mathbf{2}}\text{-}\hat{\mathbf{3}}$ ) about the vertical axis, such that,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\Omega} \times \hat{\mathbf{1}} = \Omega \hat{\mathbf{2}}, \quad \frac{d^2\hat{\mathbf{1}}}{dt^2} = \dot{\Omega} \hat{\mathbf{2}} + \Omega \boldsymbol{\Omega} \times \hat{\mathbf{2}} = -\Omega^2 \hat{\mathbf{1}} + \dot{\Omega} \hat{\mathbf{2}}, \quad \hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2} = \dot{\Omega} \hat{\mathbf{z}}. \quad (9)$$

The velocity (5) of the center of the sphere can now be written as,

$$\mathbf{v} = -\Omega(b - a) \hat{\mathbf{2}} + \dot{z} \hat{\mathbf{z}}, \quad (10)$$

so the  $\hat{\mathbf{2}}$ -component of the total angular velocity  $\boldsymbol{\omega}$  of the sphere about its center (and also that about the point of contact) is  $v_z/a = \dot{z}/a$ , and the  $\hat{\mathbf{z}}$ -component is  $v_2/a = -(b - a)/a$ . Thus,

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{1}} + \frac{\dot{z}}{a} \hat{\mathbf{2}} - \Omega \frac{b - a}{a} \hat{\mathbf{z}}, \quad \frac{d\boldsymbol{\omega}}{dt} = \dot{\omega}_1 \hat{\mathbf{1}} + \Omega \omega_1 \hat{\mathbf{2}} + \frac{\ddot{z}}{a} \hat{\mathbf{2}} - \frac{\Omega \dot{z}}{a} \hat{\mathbf{1}} - \dot{\Omega} \frac{b - a}{a} \hat{\mathbf{z}}, \quad (11)$$

With these, the equation of motion (8) becomes,

$$I \left[ \left( \dot{\omega}_1 - \frac{\Omega \dot{z}}{a} \right) \hat{\mathbf{1}} + \left( \Omega \omega_1 + \frac{\ddot{z}}{a} \right) \hat{\mathbf{2}} - \dot{\Omega} \frac{b-a}{a} \hat{\mathbf{z}} \right] = ma(b-a) \dot{\Omega} \hat{\mathbf{z}} - m(g + \ddot{z})a \hat{\mathbf{z}}, \quad (12)$$

The components of the equation of motion imply,

$$\hat{\mathbf{z}} : \quad \dot{\Omega} = 0, \quad \Omega = \text{constant}, \quad (13)$$

$$\hat{\mathbf{1}} : \quad \dot{\omega}_1 = \frac{\Omega \dot{z}}{a}, \quad \omega_1 = \frac{\Omega z}{a} + \omega_{10}, \quad (14)$$

$$\hat{\mathbf{2}} : \quad (I + ma^2) \ddot{z} + I \Omega^2 z = -ma^2 g - I \Omega \omega_{10}. \quad (15)$$

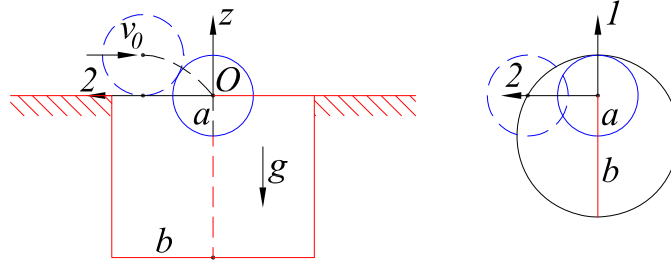
The center of the sphere executes simple harmonic motion in  $z$ ,<sup>2</sup> and if at time  $t = 0$ ,  $z = 0$ ,  $\dot{z} = \dot{z}_0$ ,  $\omega_1 = \omega_{10}$ , then,

$$z = \frac{ma^2 g + I a \Omega \omega_{10}}{I \Omega^2} (\cos \alpha t - 1) + \frac{\dot{z}_0}{\alpha} \sin \alpha t, \quad \text{where} \quad \alpha = \Omega \sqrt{\frac{I}{I + ma^2}}. \quad (16)$$

We now consider under what conditions a golf ball could roll into a cup/vertical cylinder such that at time  $t = 0$  the motion is described by eq. (16).

According to eqs. (10) and (11), the velocity  $\mathbf{v}_0$  and the angular velocity  $\boldsymbol{\omega}_0$  at this time must be,

$$\mathbf{v}_0 = -\Omega(b-a) \hat{\mathbf{2}} + \dot{z}_0 \hat{\mathbf{z}}, \quad \boldsymbol{\omega}_0 = \omega_{10} \hat{\mathbf{1}} + \frac{\dot{z}_0}{a} \hat{\mathbf{2}} - \Omega \frac{b-a}{a} \hat{\mathbf{z}}. \quad (17)$$



The figure above shows side and top views of the ball as it enters the cup, after rolling into it from the left while on the horizontal surface. At time  $t = 0$ , the ball has fallen through height  $a$ , so  $\dot{z}_0 = -\sqrt{2ag}$ . If the ball arrived at the top of the cup with horizontal velocity  $v_0$  (in the  $-\hat{\mathbf{2}}$  direction), then this is also the horizontal velocity when the center of the ball has fallen to  $z = 0$ , and so  $\Omega = v_0/(b-a)$ . The angular velocity of the ball did not change while it fell into the cup, so the angular velocity at the time of arrival was,

$$\boldsymbol{\omega}_{\text{arrival}} = \boldsymbol{\omega}_0 = \omega_{10} \hat{\mathbf{1}} - \sqrt{\frac{2g}{a}} \hat{\mathbf{2}} - \frac{v_0}{a} \hat{\mathbf{z}}, \quad \mathbf{v}_{\text{arrival}} = -v_0 \hat{\mathbf{2}} = -\Omega(b-a) \hat{\mathbf{2}}. \quad (18)$$

If the ball had been simply rolling without slipping prior to arrival at the cup, then  $\omega_{10} = v_0/a$  and the  $\hat{\mathbf{2}}$ - and  $\hat{\mathbf{z}}$ -components of  $\boldsymbol{\omega}_{\text{arrival}}$  would be zero. Hence, only under special conditions

<sup>2</sup>This motion can be regarded as a nutation about steady motion with angular velocity  $\Omega$  in a horizontal circle at  $z = -(ma^2 g + I a \Omega \omega_{10})/I \Omega^2$ .

of rolling with slipping at the moment of arrival at the cup could the ball roll into it and pop back out after following motion of the form (16).

For a golf ball of uniform mass density,  $I = 2ma^2/5$ , and  $\alpha = \sqrt{2/7}\Omega = \Omega/1.87$ . If the golf ball does pop out of the hole, it does so in somewhat less than one period of the vertical oscillation, *i.e.*, in less the 1.87 revolutions of the ball around the vertical axis of the cup.

This problem was briefly mentioned on p. 26 of Littlewood's *Miscellany* (1953),

[http://kirkmcd.princeton.edu/examples/mechanics/littlewood\\_miscellany.pdf](http://kirkmcd.princeton.edu/examples/mechanics/littlewood_miscellany.pdf)

It is solved via Lagrange's method on p. 95 of J.I. Neimark and N.A. Fufaev, *Dynamics of Nonholonomic Systems* (Am. Math. Soc., 1972),

[http://kirkmcd.princeton.edu/examples/mechanics/neimark\\_72.pdf](http://kirkmcd.princeton.edu/examples/mechanics/neimark_72.pdf)

Other discussions of it include, M. Gaultieri *et al.*, *Golfer's dilemma*, Am. J. Phys. **74**, 497 (2006), [http://kirkmcd.princeton.edu/examples/mechanics/gaultieri\\_ajp\\_74\\_497\\_06.pdf](http://kirkmcd.princeton.edu/examples/mechanics/gaultieri_ajp_74_497_06.pdf)

O. Pujol and J.P. Pérez, *On a simple formulation of the golf ball paradox*, Eur. J. Phys. **28**, 379 (2007), [http://kirkmcd.princeton.edu/examples/mechanics/pujol\\_ejp\\_28\\_379\\_07.pdf](http://kirkmcd.princeton.edu/examples/mechanics/pujol_ejp_28_379_07.pdf)

O. Pujol and J.-P. Pérez, *The golfer's curse revisited with motion constants*, Am. J. Phys. **90**, 657 (2022), [http://kirkmcd.princeton.edu/examples/mechanics/pujol\\_ajp\\_90\\_657\\_22.pdf](http://kirkmcd.princeton.edu/examples/mechanics/pujol_ajp_90_657_22.pdf)