A Relativistic Electron Can't Extract Net Energy from a Long Laser Pulse

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In this note I write out the argument of Sprangle *et al.* [1] at some length – and add to it a result, eq. (9), that I have not seen published elsewhere. The specific point of this argument is that the longitudinal component of the laser field near a focus, neglected in the analysis of [2], cancels any energy transferred to a relativistic electron by the transverse component.[1]

The Gaussian approximation to a laser beam that propagates along the $+z$ axis and is polarized in the $\mathbf{x}$ direction is, to leading order,

$$\mathbf{E} = \frac{E_0\hat{\mathbf{x}}}{\sqrt{1+\varsigma^2}} e^{-i\tan^{-1}\varsigma} e^{i\rho^2\varsigma/(1+\varsigma^2)} e^{-\rho^2/(1+\varsigma^2)} e^{i\varphi}, \tag{1}$$

where $\rho^2 = (x^2+y^2)/w_0^2$, the radius of the waist is w_0, $\varsigma = z/z_0$, the Rayleigh range is,

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}, \tag{2}$$

the phase is,

$$\varphi = kz - \omega t, \tag{3}$$

the frequency of the wave is ω, the wave number is $k = \omega/c$, the wavelength is λ and c is the speed of light.

Expression (1) describes the main features of the focus of the laser beam, but it does not satisfy the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$. To see this, note that a divergence-free field that points only in the x direction cannot vary with x.

We find below that an electric field which satisfies Maxwell's equations in the next approximation has a small longitudinal component, and that this apparently small component is sufficient to cancel completely any net energy transfer to a relativistic electron that appears possible if only eq. (1) is used.

Equation (1) describes a continuous-wave laser beam, rather than a pulse.

Equation (1) of [2] is the same as my eq. (1) with the addition of a factor $g(\varphi)$ that describes the laser pulse envelope in time. The new part of what follows is to find a condition on the form of g so that (1) satisfies Maxwell's equations to a good approximation. I will find that the assumption in [2] that $g = \sin^2(\varphi/\varphi_0)$ for $0 < \varphi/\varphi_0 < \pi$ and zero elsewhere is not satisfactory. This oversight is in addition to the erroneous claim in [2] that the longitudinal part of field $\mathbf{E}$ can be neglected.

Before I deal with the issue of acceleration of electrons, I review the derivation of a better approximation to a Gaussian laser pulse, following [3]. The knowledgeable reader might want to skip ahead to eqs. (26) and (27).

[1]A small acceleration of electrons "in vacuum" by an intense laser was reported in [2], but this is a subtle effect, not explained in the first approximation presented there.

A key insight of [3] is that the form of eq. (1) can more properly be used for the vector potential **A** than for the electric field **E**. In general, the divergence of the vector potential need not be zero and there are solutions to Maxwell's equations with nonuniform vector potential that point only along the x-axis.

A second useful insight is that when the wave equation for the vector potential is written in terms of the dimensionless variables,

$$\xi = \frac{x}{w_0}, \quad \upsilon = \frac{y}{w_0}, \quad \rho^2 = \xi^2 + \upsilon^2, \quad \text{and} \quad \varsigma = \frac{z}{z_0}, \tag{4}$$

then a series expansion suggests itself. Namely, the focal region has transverse and longitudinal extent in the ratio,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}. \tag{5}$$

The aspect ratio θ_0 (also the diffraction angle) is typically much less than one, and so can serve as the expansion parameter.

We seek fields that propagate in the $+z$ direction, have limited transverse extent, and for which the vector potential has only an x component. We try,

$$\mathbf{A}(\mathbf{r}, t) = \hat{\mathbf{x}}\psi(\mathbf{r})g(\varphi)\, e^{i\varphi}, \tag{6}$$

where ψ and g vary "slowly." The vector potential must satisfy the free-space wave equation,

$$\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{7}$$

Inserting trial solution (6) into (7) we find that,

$$\nabla^2 \psi + 2ik\frac{\partial \psi}{\partial z}\left(1 - \frac{ig'}{g}\right) = 0, \tag{8}$$

where $g' = dg/d\varphi$. Since ψ is a function of $\mathbf{r}$ while g and g' are functions of the phase φ, eq. (8) cannot be satisfied in general. Often the discussion is restricted to the case where $g' = 0$, *i.e.*, to continuous waves. However, we see that to proceed with our description of pulsed beams we must accept the condition that,

$$g' \ll g. \tag{9}$$

First, consider the proposal of [2] that $g = \sin^2(\varphi/\varphi_0)$. Then, $g'/g = (2/\varphi_0)\cot(\varphi/\varphi_0)$. Even for the plausible restriction that $\varphi_0 \gg 2$, g'/g blows up at the beginning and end of the pulse (which is defined on the interval $0 \le \varphi \le \pi\varphi_0$).

Another popular form for a laser pulse is a Gaussian: $g = \exp[-(\varphi/\varphi_0)^2]$. Then, $g'/g = -2\varphi/\varphi_0^2$ which does not satisfy (9) for $|\varphi| \gtrsim \varphi_0$.

A more appropriate form for a pulsed beam is a hyperbolic secant (as arises in studies of solitons[2]),

$$g(\varphi) = \operatorname{sech}\left(\frac{\varphi}{\varphi_0}\right). \tag{10}$$

[2]See, for example, [4].

Then, $g'/g = -(1/\varphi_0)\tanh(\varphi/\varphi_0)$, which is much less than one everywhere for $\varphi_0 \gg 1$.

One might hope that a poor approximation would suffice in regions where the field is weak. But, when the field is probed by a moving electron, the latter spends a long time in the weak-field region, so time integrals such as energy transfer have significant contributions from that region.

Hence, I strongly recommend that future numerical (and analytic) calculations involving laser pulses use form (10).

We now suppose that condition (9) is satisfied, so that (8) can be approximated as,

$$\nabla_\perp^2 \psi + 4i\frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \tag{11}$$

where,

$$\nabla_\perp^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \upsilon^2}, \tag{12}$$

in terms of the dimensionless variables introduced in eqs. (4-5). This form suggests the series expansion,

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + ... \tag{13}$$

Inserting this into eq. (11) and collecting terms of order θ_0^0 and θ_0^2, we find,

$$\nabla_\perp^2 \psi_0 + 4i\frac{\partial \psi_0}{\partial \varsigma} = 0, \tag{14}$$

and,

$$\nabla_\perp^2 \psi_2 + 4i\frac{\partial \psi_2}{\partial \varsigma} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}, \tag{15}$$

respectively. Equation (14) can be recognized as the paraxial wave equation whose Gaussian solution was given in eq. (1). That is,

$$\psi_0 = f\, e^{-f\rho^2}, \tag{16}$$

where,

$$f = \frac{-i}{\varsigma - i} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1+\varsigma^2}}. \tag{17}$$

In [3], the solution to eq. (15) was cleverly guessed,

$$\psi_2 = \left(\frac{f}{2} - \frac{f^3\rho^4}{4}\right)\psi_0, \tag{18}$$

although we will not need this here.

We work in the Lorentz gauge (and Gaussian units), so the scalar potential ϕ obeys,

$$\frac{\partial \phi}{\partial t} = -c\,\boldsymbol{\nabla}\cdot\mathbf{A}. \tag{19}$$

Similarly to eq. (6), we suppose that ϕ can be written as,

$$\phi(\mathbf{r}, t) = \Phi(\mathbf{r})g(\varphi)\, e^{i\varphi}. \tag{20}$$

Then,

$$\frac{\partial \phi}{\partial t} = -\omega\phi \left(1 - \frac{ig'}{g}\right) \approx -\omega\phi, \tag{21}$$

when condition (9) is satisfied. In this case,

$$\phi = -\frac{i}{k}\nabla \cdot \mathbf{A}. \tag{22}$$

The electric and magnetic fields can now be deduced from the approximate vector potential,

$$\mathbf{A} = \psi_0\, g(\varphi)\, e^{i\varphi}\, \hat{\mathbf{x}}, \tag{23}$$

via,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + ik\mathbf{A}, \tag{24}$$

and,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{25}$$

again approximating as one the factors $1 - ig'/g$ that arise. The results accurate to order θ_0 are, after dividing out a factor of ik,

$$\begin{aligned} E_x &= \psi_0\, g\, e^{i\varphi}, \\ E_y &= 0, \\ E_z &= \frac{i\,\theta_0}{2}\frac{\partial \psi_0}{\partial \xi} g\, e^{i\varphi} = -i\,\theta_0\, f\, \xi\, E_x, \end{aligned} \tag{26}$$

$$\begin{aligned} B_x &= 0, \\ B_y &= E_x, \\ B_z &= \frac{i\,\theta_0}{2}\frac{\partial \psi_0}{\partial \upsilon} g\, e^{i\varphi} = -i\,\theta_0\, f\, \upsilon\, E_x. \end{aligned} \tag{27}$$

These expression satisfy $\boldsymbol{\nabla} \cdot \mathbf{E} = 0 = \boldsymbol{\nabla} \cdot \mathbf{B}$ plus terms of order θ_0^2.

After these lengthy preliminaries we are ready to consider vacuum laser acceleration of electrons.

We consider an electron moving at velocity c along the line $x = z\theta$, where θ is a small angle. The electron passes through a laser field given by eqs. (26-27), with $y = 0$ always, and we suppose that it passes the origin at $t = 0$. In terms of the dimensionless variables ξ, ς and ρ, the trajectory is,

$$\rho = \xi = \frac{\theta\varsigma}{\theta_0}. \tag{28}$$

The electric field component along the electron's trajectory is,

$$E_\parallel = E_x \sin\theta + E_z \cos\theta \approx E_x\, \theta(1 - i\,f\,\varsigma) = E_x\, \theta\, f, \tag{29}$$

for small θ, noting that eqs. (26) and (28) lead to,

$$E_z = -i\,\theta_0\, f\, \xi\, E_x = -i\,\theta\, f\, \varsigma\, E_x, \tag{30}$$

and that eq. (17) leads to the identity,

$$1 - i f \varsigma = f. \tag{31}$$

The electron has coordinate z at time $t = z/(c\cos\theta)$ so,

$$\varphi = kz - \omega t \approx -\frac{k\,z\,\theta^2}{2} = -\frac{\theta^2}{\theta_0^2}\,\varsigma. \tag{32}$$

Then,

$$\begin{aligned} E_x &= E_0\,f\,g\,e^{-f\rho^2}\,e^{i\varphi} \\ &\approx E_0\,f\,g\,e^{-f\varsigma^2\theta^2/\theta_0^2}\,e^{-i\,\varsigma\,\theta^2/\theta_0^2} = E_0\,f\,g\,e^{-i\,f\,\varsigma\,\theta^2/\theta_0^2}, \end{aligned} \tag{33}$$

using eqs. (26), (31) and (32). Inserting this into eq. (29) we have,

$$\begin{aligned} E_\parallel &= \theta\,E_0\,f^2\,g\,e^{-i\,f\,\varsigma\,\theta^2/\theta_0^2} \\ &= \frac{d}{d\varsigma}\left(\frac{i\,\theta_0^2}{\theta}E_0\,g\,e^{-if\,\varsigma\,\theta^2/\theta_0^2}\right), \end{aligned} \tag{34}$$

again neglecting a term in g'/g. That is, the force on the electron along its trajectory can be derived from a potential. The change in energy along the trajectory is then just the change in the potential. However, the potential,

$$U = -\frac{i\,\theta_0^2}{\theta}E_0\,g\,e^{-i\,f\,\varsigma\,\theta^2/\theta_0^2} \tag{35}$$

has the same value at $\varsigma = -\infty$ and $+\infty$, so the (relativistic) electron gains no net energy as it crosses the laser beam.

This is the argument of [1], with the addition that it holds for laser pulses as well as for continuous waves, so long as condition (9) is satisfied – which condition is required if (26) and (27) are to be (approximate) solutions to Maxwell's equations.

Everyone agrees that if one uses E_x from (26) but ignores E_z, a net energy gain will be calculated for a free electron crossing a laser beam. See also [5].

The interesting case of extremely short pulses ($g'/g \approx 1$) has been considered in [6]. The above analysis does not hold in this limit, and significant energy can in principle be transferred from a single-cycle pulse to an electron. However, for a pulse of even a few cycles, the energy transfer is greatly suppressed.

References

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