FitzGerald's Calculation of the Radiation of an Oscillating Magnetic Dipole

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1 Problem

Without using Poynting's theorem [1], deduce the power radiated by a small loop of current, $I(t) = I_0\, e^{-i\omega t}$, that oscillates with angular frequency ω, where the radius a of the loop obeys $ka \ll 1$, where $k = \omega c = 2\pi/\lambda$ and c is the speed of light in vacuum, which surrounds the loop.

2 Solution

This problem was first solved by FitzGerald in 1883 [2], one month before Poynting introduced his vector [1], and several years before Hertz [3] calculated the power radiated by a small, oscillating electric dipole (using the Poynting vector).[1]

The radiated power flows outwards from the source at the speed of light. Hence, we can relate the (time-average) radiated power P to the (time-average) density $u(r, \theta, \phi)$ of the energy of the electromagnetic radiation by,

$$\frac{dP}{d\Omega} = cr^2 u, \tag{1}$$

for distances r large compared to the size of the source, using a spherical coordinate system (r, θ, ϕ) centered on the source, where $d\Omega = d\cos\theta\, d\phi$ is an element of solid angle with respect to the origin (at the center of the source). Following Maxwell [10], we understand that the energy density u (in vacuum) can be written (in Gaussian units) in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ as,

$$u = \frac{E^2 + B^2}{8\pi}\,. \tag{2}$$

Although FitzGerald must have been aware of this relation, he did not use it. Rather, his argument was based on the potentials of the electromagnetic fields.

[1]A study by Lamb (1883) [4] of the electrical oscillations of a spherical conductor emphasized the interior fields, with little interest in exterior fields, where the speed of propagation was taken to be infinite. In 1884, J.J. Thomson noted that the exterior fields of a spherical conductor with an initial nonuniform charge distribution would die out very quickly due to *energy be propagated from the shell into the surrounding dielectric*, which was considered to be largely uninteresting. Rowland (1884) [6] gave a general discussion of spherical electromagnetic solutions to the Helmholtz wave equation, in the context of scattering of light by spheres. The possibility of electric dipole radiation from *two spherical conductors connected to an alternating dynamo machine* was discussed qualitatively by W. Thomson in his 1884 Baltimore Lectures, p. 44 of [7] and briefly noted on p. 463 of [8]. In 1888, Heaviside [9] discussed forced electrical oscillation of spherical conductors, arriving in sec. 27 to some understanding of electric dipole radiation, which he considered to be a *waste* of energy.

The present problem concerns electrical currents, which can be taken to flow in a circuit where the electric charge density is everywhere zero. Then, the electric scalar potential V is everywhere zero.

The vector potential $\mathbf{A}$ is, however, nonzero. So, we can consider a relation between the energy density and the vector potential. Such a relation was given by Maxwell in eq. (37) of his great paper of 1864 [10], and in eq. (16) of Art. 634 of his *Treatise* [11],

$$u = \frac{\mathbf{J} \cdot \mathbf{A}}{2c} . \tag{3}$$

Maxwell's equation (3) was deduced for a steady current distribution $\mathbf{J}$, which would now be called a conduction current density. In time-dependent situations, besides the conduction current density $\mathbf{J}_{\rm conduction}$ there also exists the so-called displacement current density,

$$\mathbf{J}_{\rm displacement} = \frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t} \to \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} , \tag{4}$$

where the latter form holds in vacuum (as considered in this note). FitzGerald [2] applied Maxwell's "static" equation (3) to the present dynamic example, using $\mathbf{J} = \mathbf{J}_{\rm conduction} + \mathbf{J}_{\rm displacement}$ without comment or reference to Maxwell. In this section we accept FitzGerald's conjecture, and comment on its validity in sec. 3.

In general, the electric field is related to the potentials according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} . \tag{5}$$

In examples such as the present where the electric charge density is zero, and consequently $V = 0$, the displacement current is a function only of the vector potential,

$$\mathbf{J}_{\rm displacement} = -\frac{1}{4\pi c}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\omega^2 \mathbf{A}}{4\pi c} , \tag{6}$$

where the latter form holds for when the currents depend only on a single angular frequency ω. Combining eqs. (3) and (6), the (time-average) energy density is (at large distances where the quasistatic magnetic-field energy of $\mathbf{J}_{\rm conduction}$ can be neglected),

$$u = \frac{\omega^2 \left|A\right|^2}{16\pi c^2} = \frac{k^2 \left|A\right|^2}{16\pi} . \tag{7}$$

FitzGerald follows Faraday and Maxwell in identifying the energy density (7) as a kind of "kinetic" energy of the ether. In addition, they considered the electrical energy of the system (which they called the "potential" energy). In a footnote [2], FitzGerald remarks that the time-average "kinetic" and "potential" energies should be equal, and so the total, time-average energy density should be twice that of eq. (7),

$$u_{\rm total} = \frac{k^2 \left|A\right|^2}{8\pi} . \tag{8}$$

Then, the (time-average) angular distribution of radiated power is, from eq. (1),

$$\frac{dP}{d\Omega} = c r^2 u_{\rm total} = \frac{c k^2 r^2 \left|A\right|^2}{8\pi} . \tag{9}$$

All that remains is to find the vector potential of the oscillating current.

The current distribution $\mathbf{J}_{\text{conduction}}$ flows in a loop of radius a in the x-y plane, centered on the origin, and is independent of azimuth ϕ when $a \ll \lambda$ ($ka \ll 1$) as assumed here. Then, the vector potential has only a ϕ component, and is azimuthally symmetric.

Following Lorenz [12] (1867), the vector potential observed at $(r, \theta, 0)$ can be calculated from the retarded current density,

$$A_\phi(r, \theta, 0, t) = \int \frac{J_{\phi'}(r', t' = t - R/c)\cos\phi'}{cR}\, d\text{Vol}', \tag{10}$$

where the distance R from the source point to the observer at $r \gg a$ is,

$$\begin{aligned} R &= \sqrt{(r\sin\theta - a\cos\phi')^2 + (a\sin\phi')^2 + (r\cos\theta)^2} = \sqrt{r^2 - 2ar\sin\theta\cos\phi' + a^2} \\ &\approx r - a\sin\theta\cos\phi'. \end{aligned} \tag{11}$$

For an oscillating current in the small loop of the form $I_0\, e^{-i\omega t}$ we can write,

$$\begin{aligned} A_\phi(r, \theta, 0, t) &\approx \int_0^{2\pi} \frac{I_0\, e^{-i\omega(t-R/c)}\cos\phi'}{cR}\, a\, d\phi' \approx \frac{I_0\, a}{cr}\int_0^{2\pi} e^{-i\omega t + i\omega r/c - i\omega a\sin\theta\cos\phi'/c)}\cos\phi'\, d\phi' \\ &\approx \frac{I_0\, a\, e^{i(kr-\omega t)}}{cr}\int_0^{2\pi}(1 - ika\sin\theta\cos\phi')\cos\phi'\, d\phi' = -i\pi a^2 k I_0 \sin\theta \frac{e^{i(kr-\omega t)}}{cr} \\ &= -ikm\sin\theta\frac{e^{i(kr-\omega t)}}{r}\,, \end{aligned} \tag{12}$$

where $m = \pi a^2 I_0/c$ is the magnetic moment of the current loop.

However, in 1883 the retarded potentials were not well known, perhaps due to Maxwell's apparent distrust of them [13]. It seems that FitzGerald independently (re)invented the retarded potentials [2].[2]

Using the approximation (12) in eq. (9) we obtain the time-average angular distribution of power radiated by the small oscillating magnetic dipole,

$$\frac{dP}{d\Omega} \approx \frac{ck^4 m^2 \sin^2\theta}{8\pi} = \frac{\pi(ka)^4 I_0^2 \sin^2\theta}{8c}\,, \tag{13}$$

whose integral is,

$$P \approx \frac{\pi^2(ka)^4 I_0^2}{3c} = \frac{1}{2}\frac{\pi}{6}\frac{4\pi}{c}\left(\frac{L}{\lambda}\right)^4 I_0^2 \equiv \frac{R_{\text{rad}} I_0^2}{2}\,, \tag{14}$$

where the so-called radiation resistance of a small loop of circumference $L = 2\pi a$ is,

$$R_{\text{rad}} = \frac{\pi}{6}\frac{4\pi}{c}\left(\frac{L}{\lambda}\right)^4 = 197\left(\frac{L}{\lambda}\right)^4\ \Omega, \tag{15}$$

[2] A retarded velocity potential for acoustics was described by Rayleigh in 1877 [14], which likely influenced FitzGerald's derivation of the retarded vector potential. Thanks to B.J. Hunt for pointing this out. See also [15]. FitzGerald elaborated further on retarded potentials in [16]; see [17, 18, 19] for his thinking prior to [2].

noting that $4\pi/c = 377\,\Omega$ (= "resistance of the vacuum").

An implication of eq. (14) is that a DC current ($\omega = kc = 0$) doesn't radiate.[3]

3 Validity of FitzGerald's Conjectures

FitzGerald might have argued that,

$$U_M = \int \frac{\mathbf{B}\cdot\mathbf{H}}{8\pi}\, d\mathrm{Vol} = \int \frac{\mathbf{H}\cdot\boldsymbol{\nabla}\times\mathbf{A}}{8\pi}\, d\mathrm{Vol} \approx \int \frac{\mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{H}}{8\pi}\, d\mathrm{Vol}$$
$$= \int \frac{\mathbf{A}\cdot[\mathbf{J}_{\mathrm{cond}} + (1/4\pi)\partial\mathbf{D}/\partial t]}{2c}\, d\mathrm{Vol}, \qquad (16)$$

where the approximation requires the neglect of the surface integral at infinity, $\int \mathbf{A}\cdot\mathbf{H}\, d\mathrm{Area}$, that arises in the integration by parts in eq. (16). The vector potential $\mathbf{A}$ of an oscillating magnetic dipole is purely azimuthal as its conduction currents are purely azimuthal, so the magnetic field (in vacuum), $\mathbf{H} = \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, has no azimuthal component, and the surface integral vanishes for this example. But in general, eq. (16) does not hold when radiation is present.

FitzGerald obtained correct results (13)-(14) for the power radiated by a small oscillating current loop via a brilliant conjecture that the time-average "kinetic" energy density of the radiation could be related to the vector potential by eq. (7), and that the time-average "potential" energy density of the radiation has the same value.[4] The latter relation is indeed true in general (for radiation in vacuum), but eq. (7) is valid only if the electric field is entirely due to the vector potential, which implies that the electric charge density (and the scalar potential in the Lorenz gauge) must vanish everywhere. This requirement is met for currents oscillating in loops if the characteristic size of the loop is small compared to a wavelength, but it is not true in general. Hence, FitzGerald's method is appropriate for the problem that he considered, but does not provide a method of analysis of general oscillating current distributions.

In particular, distributions characterized by oscillating electric-dipole moments cannot be so analyzed. Only after Poynting [1] provided a more general view of energy density and flow in electromagnetic fields could more general radiation problems be successfully analyzed, starting with the great work of Hertz [3].

Furthermore, FitzGerald's analysis does not provide an accurate assessment of the nonradiative (sometimes called "reactive") energy density in the near field of the oscillating current loop, where the densities of electric and magnetic energy are very different.[5]

[3]The present computation is performed in the dipole approximation, so one might worry that radiation exists due to effects of higher-order multipoles. For an argument that this not so, see [20].

A different view is that radiation involves photons with nonzero frequency, but a DC current has no nonzero frequency content, and hence no radiation. This type of argument is reviewed in [21].

[4]When $V = 0$, then $\mathbf{E} = k\mathbf{A}$, and the energy density (7) equals $|E|^2/16\pi$, which we recognize as the time-average electric rather than magnetic energy density, despite the origin of eq. (7) in the magnetostatic relation (3). Such complexities dramatize the contribution of Poynting [1] in clarifying the relation of the electric and magnetic energy densities to the flow of energy in the electromagnetic field.

[5]See, for example, [22].

Finally, it is interesting to note that FitzGerald regarded an oscillating current loop as a possible model for optical radiation by an atom. He understood that atoms are smaller than optical wavelengths, so his magnetic-dipole radiation would be a very weak effect. He does not seem to have appreciated that such radiation by a classical atom would quickly lead to its collapse.[6]

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