Electromagnetic-Field Angular Momentum of a Classical Charged Particle in a Uniform Magnetic Field

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1 Problem

What is the electromagnetic-field angular momentum of a charged particle in a circular orbit in a uniform magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, supposing the velocity of the particle is small compared to the speed c of light?

This problem was inspired by [1], where it was claimed (after eq. (19)) that the electric field of the circling charge q obeys $\boldsymbol{\nabla} \times \mathbf{E}_q = 0$, which implies that the magnetic field of the charge is constant in time. This is not so,[1] *and hence a better solution for the field angular momentum is required.*[2]

2 Solution

This problem is somewhat ill posed, as a charged particle in a circular orbit radiates angular momentum, such that the angular momentum stored in the electromagnetic field depends on the past history of the particle.

To give the discussion a crisper basis, we suppose an electric charge q of rest mass m is initially at rest in the magnetic field, and is slowly accelerated to velocity $\mathbf{v}_0$ (with $v_0 \ll c$) along a straight line perpendicular to $\mathbf{B}_0$, and released at time $t = 0$. We ignore the tiny amount of radiation emitted during the initial acceleration,[3] such that lines of the magnetic field $\mathbf{B}_q(t = 0)$ of the charged particle are circles (rather than a vortex) in planes perpendicular to $\mathbf{v_0}$. Then, at time $t = 0$ the interaction field energy, $U_{\rm EM,0} = \int \mathbf{B}_q(t = 0) \cdot \mathbf{B}_0\, d{\rm Vol}/4\pi$ (in Gaussian units) is zero, and the total energy of the charged particle is just its kinetic energy $U_0 = mv_0^2/2$.

[1] In [1], "quantum" particles are considered, whose stationary states in a uniform magnetic field can be a recently discovered [2, 3] variant of Landau levels (for the latter, see, for example, §111, p. 424 of [4]) called electron-vortex states. The charge density ρ and current density $\mathbf{J}$ associated with these electron-vortex states are time independent, such that one can say that the electric and magnetic fields of the charged particle are time independent, and hence $\boldsymbol{\nabla} \times \mathbf{E}_q = 0$ while these states last. However, the vortex states (except the lowest-energy level) decay via photon emission, so that these states do have an association with time-dependent electric fields.

Here, we consider this association in a "classical" context, where circling charged particles emit radiation at all times (in contrast to the "quantum" case where radiation is emitted only during "quantum jumps").

[2] A famous example of (classical) field angular momentum was given by Feynman [5]. See also [6] and references therein.

[3] The rate of radiation of energy (and angular momentum) varies as a^2 according to the Larmor formula (3), where the initial acceleration is $a = v_0/T$ for constant acceleration during time interval T. The total angular momentum radiated at $t < 0$ varies as $a^2 T \propto 1/T$, which is negligible for large T.

At time $t = 0$ the charged particle is set free, and enters a circular orbit of radius $r(t = 0) \equiv r_0$, losing energy slowly due to electromagnetic radiation, such that $r(t)$ drops to zero. The charge eventually comes to rest at the center of its circular orbit at time $t = 0$.

The angular momentum about the center of the circular orbit of radius r_0 (at time $t = 0$) is simply the mechanical angular momentum, $L(t = 0) = mv_0 r_0$, of the charge. That is, the field angular momentum, $\mathbf{L}_{\rm EM}(t = 0) = \int \mathbf{r}' \times (\mathbf{E}_q(t = 0) \times \mathbf{B}_0)\, d{\rm Vol}'/4\pi c$, is zero, taking the self-field angular momentum of the charge to be part of its mechanical angular momentum, and noting that the electric field $\mathbf{E}_q(t = 0)$ is essentially the (spherically symmetric) static electric field of the charge at that time.

For $v_0 \ll c$, trajectory of the particle as always approximately circular, so the speed $v(t)$ is related to the radius $r(t)$ of the circular orbit in the x-y plane by,

$$F = m\frac{v^2}{r} = \frac{qvB_0}{c} \qquad v = \frac{qB_0 r}{mc}\,, \qquad r_0 = \frac{mcv_0}{qB_0}\,, \tag{1}$$

and the (radial) acceleration is,

$$a = \frac{v^2}{r} = \frac{q^2 B_0^2 r}{m^2 c^2}\,. \tag{2}$$

The power radiated as the charge spirals inward to the center of the circular orbit follows from the Larmor formula,

$$\frac{dU}{dt} = -\frac{2q^2 a^2}{3c^3} = -\frac{2q^6 B_0^4 r^2}{3m^4 c^7}\,. \tag{3}$$

We approximate the total energy of the spiraling charged particle as its kinetic energy,

$$U \approx \frac{mv^2}{2} = \frac{q^2 B_0^2 r^2}{2mc^2}\,, \qquad \frac{dU}{dt} \approx \frac{q^2 B_0^2 r}{mc^2}\frac{dr}{dt}\,. \tag{4}$$

Then, the equation of motion of the spiral is,

$$\frac{dr}{dt} \approx -\frac{2q^4 B_0^2 r}{3m^3 c^2} \quad (\ll v), \qquad r = r_0\, e^{-2q^4 B_0^2 t/3m^3 c^5}. \tag{5}$$

The mechanical angular momentum goes to zero with r,

$$L_{\rm mech} = mvr = \frac{qB_0 r^2}{c} \tag{6}$$

so conservation of angular momentum implies that the field angular momentum obeys,

$$L_{\rm EM}(t \geq 0) = L_{\rm mech}(0) - L_{\rm mech}(t) \approx \frac{qB_0 r_0^2}{c}\left(1 - e^{-4q^4 B_0^2 t/3m^3 c^5}\right), \tag{7}$$

recalling that $\mathbf{L}_{\rm EM,0} \approx 0$. The final field angular momentum is entirely in the radiation fields of the charge.

References

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