Constant Acceleration and the Equivalence Principle

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1 Problem

A common statement of the Equivalence Principle is that physics inside a box in a uniform gravitational field of 1 g, such as that at the surface of the Earth, is equivalent to that inside a box in zero gravitational field that is accelerating at 1 g.

However, if the box accelerated at 1 g ≈ 9.8 m/s^2 for 1 year ($\approx \pi \times 10^7$ s), its speed would be $v = gt \approx 9.8 \times \pi \times 10^7 \approx 3 \times 10^8$ m/s, which is the speed c of light. But, a box cannot go faster than the speed of light, so this suggests that the Equivalence Principle holds only for one year, after which it is invalid.

Does this argument really show that the Equivalence Principle is bogus?

2 Solution

The key issue is that for an acceleration of the box to be equivalent to a gravitational field acting on it, the constant acceleration must be relative to the box, rather than, say, to an inertial observer outside the box.

Constant acceleration of the box relative to such an inertial observer would not seem constant relative to an observer inside the box. And, constant acceleration of the box relative to the box itself would not seem constant relative to an inertial observer outside the box.

Acceleration, like velocity, is a relative concept.[1]

We now go into some details to support the above statements.

2.1 "Uniformly" Accelerated Motion with Respect to an Inertial Frame

"Uniform" acceleration of an object relative to an inertial lab frame does not mean constant acceleration a, as this would imply faster-than-light motion of the object for time $t > c/a$, assuming the object started from rest at $t = 0$. The constant acceleration of the object would cease after this time, so this acceleration could not be called "uniform".

Rather, we suppose (following Born [2]) that the acceleration is uniform with respect to the instantaneous inertial rest frame of the accelerated object. Quantities in this frame will be designated with the superscript $\star$. From sec. 10 of Einstein's first paper on relativity [3] we have that for acceleration parallel to the velocity $\mathbf{v}$ of an object, the acceleration $a = dv/dt$

[1] Acceleration is independent of the observer in Galilean relativity. Additional discussion of this theme is given in [1].

in the inertial lab frame is related to that, $a^\star = dv^\star/dt^\star$, in the instantaneous inertial rest frame according to,

$$a = \frac{dv}{dt} = (1 - v^2/c^2)^{3/2}\frac{dv^\star}{dt^\star} = \frac{a^\star}{\gamma^3}\,, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{1}$$

In this, two powers of $\sqrt{1 - v^2/c^2}$ come from the transformation of relative velocity, and another comes from time dilation.

For uniform acceleration, $a^\star$ = constant (not a = constant), eq. (1) can be integrated to find the velocity v. Then, the acceleration a in the lab frame is related to that in the instantaneous inertial rest frame of the box according to,

$$\frac{v}{\sqrt{1 - v^2/c^2}} = a^\star t, \qquad v = \frac{dz}{dt} = \frac{a^\star t}{\sqrt{1 + a^{\star 2}t^2/c^2}}\,, \qquad a = \frac{dv}{dt} = \frac{a^\star(1 - v^2/c^2)}{\sqrt{1 + a^{\star 2}t^2/c^2}}\,, \tag{2}$$

supposing that $v = 0$ when $t = 0$. As $t \to \infty$, $v \to c$. Note that the acceleration a in the inertial lab frame decreases with time (for $t > 0$), although the acceleration $a^\star$ in the instantaneous inertial rest frame of the box is constant.

Integrating eq. (2) we obtain,

$$z = z_0 + \frac{c^2}{a^\star}\left(\sqrt{1 + a^{\star 2}t^2/c^2} - 1\right), \tag{3}$$

where z_0 is the z-coordinate of the box at time $t = 0$. The (proper) time $t^\star$ on a clock carried inside the accelerating box is related by,

$$dt^\star = dt\sqrt{1 - v^2/c^2} = \frac{dt}{\sqrt{1 + a^{\star 2}t^2/c^2}}\,, \tag{4}$$

and hence,

$$t^\star = \frac{c}{a^\star}\sinh^{-1}\frac{a^\star t}{c}\,, \qquad t = \frac{c}{a^\star}\sinh\frac{at^\star}{c}\,. \tag{5}$$

Using this, eqs. (2) and (3) can be rewritten as,

$$v = c\,\tanh\frac{a^\star t^\star}{c}\,, \qquad \text{and} \qquad z = z_0 + \frac{c^2}{a^\star}\left(\cosh\frac{a^\star t^\star}{c} - 1\right). \tag{6}$$

As such, uniformly accelerated motion is often called "hyperbolic motion".[2,3]

Finally, we note that for times such that $|a^\star t| \ll c$, the position is well approximated by the Newtonian form,

$$z \approx Z + \frac{a^\star t^2}{2} \qquad (|a^\star t| \ll c). \tag{7}$$

[2]Hyperbolic motion appears to have been first discussed briefly by Minkowski [4], and then more fully by Born [2] and Sommerfeld [5].

[3]An object that extends from z_1 to z_2 when at rest at time $t = 0$ has extent $|z_2 - z_1|$ at all other times when all points in the object are subject to the same, uniform acceleration; there is no Lorentz contraction according to lab-frame observers for this type of uniform acceleration of an extended object.

2.2 Constant Acceleration with Respect to the Inertial Lab Frame

We again consider a box that is somehow accelerated with respect to the inertial lab frame with constant acceleration $a = g\,\hat{\mathbf{z}}$, starting from rest at the origin at time $t = 0$. Then, with respect to the lab frame,

$$v = gt, \qquad z = gt^2/2\,, \qquad \gamma(t) = \frac{1}{\sqrt{1-(gt/c)^2}} \qquad (0 < t < c/g)\,. \tag{8}$$

These relations hold only for times $0 < t < c/g$, as the the box would reach the speed of light at time $t = g/c$, and have infinite kinetic energy with respect to the lab frame, which would require an infinite energy from the source of the acceleration.

In the instantaneous inertial rest frame (the $\star$ frame) of the box at lab-frame time $0 < t < c/g$, the acceleration is,

$$a^\star = \gamma^3(t)g, \tag{9}$$

recalling eq. (1). As $t \to c/g$ in the inertial lab frame, the acceleration $a^\star$ in the instantaneous rest frame of the box goes to infinity, as does the energy required to maintain this acceleration.

If $a = g = 980$ cm/s^2, then the time $t = c/g$ at which the box's velocity reaches c in the inertial lab frame is approximately 1 Earth year, and the distance $gt^2/2 = ct/2$ that the box has traveled in this frame is approximately 1/2 light year.

The apparent weight of, say, a person of mass m inside the box is $ma^\star = \gamma^3(t)mg$, which increases with time. The person would soon feel his weight to be increasing, and would know that the box is not in a uniform gravitational field.

A Appendix: Uniformly Accelerated Reference Frame

A set of uniformly accelerated observers can be used to define a uniformly accelerated reference frame. However, the distance between observers in a "rigid" frame must be independent of time in that frame.[4] If we use the set of observers with equal spacing in the inertial lab frame at all times during their accelerated motion according to eq. (6), the distance between observers would vary with time in the accelerated frame.[5]

An appropriate coordinate system (x', y', z', t') for a "rigid" frame whose origin has acceleration g with the respect to the z-axis of the inertial lab frame is defined by eq. (140), sec. 97 of [8],[6]

$$x = x', \tag{11}$$

$$y = y', \tag{12}$$

$$z = \left(z' + \frac{c^2}{g}\right) \cosh \frac{g\, ct'}{c^2} - \frac{c^2}{g}\,, \tag{13}$$

$$ct = \left(z' + \frac{c^2}{g}\right) \sinh \frac{g\, ct'}{c^2}\,. \tag{14}$$

It is useful to note that according to eqs. (13)-(14),

$$\cosh \frac{gt'}{c} = \sqrt{1 + \left(\frac{gct}{gz' + c^2}\right)^2}, \tag{15}$$

$$z = \frac{gz' + c^2}{g} \sqrt{1 + \left(\frac{gct}{gz' + c^2}\right)^2} - \frac{c^2}{g}\,, \tag{16}$$

from which we obtain the velocity v in the lab frame of a point at constant z' in the accelerated frame as,

$$v = \frac{dz}{dt} = \frac{gc^2 t}{(gz' + c^2)\sqrt{1 + \left(\frac{gct}{gz'+c^2}\right)^2}} = c \tanh \frac{gt'}{c}\,. \tag{17}$$

[4] In general relativity, any definition of a reference frame is valid, and a set of uniformly accelerated observers with equal spacings at all times in the inertial lab frame does define a reference frame, which is preferred by some authors, as in [6]. Here, we work in the spirit of special relativity, where reference frames are "rigid".

[5] Einstein missed this issue in his first discussion of accelerated motion in 1907 [7], as he only considered accelerated observers with velocities, relative to an inertial frame, that were small compared to the speed of light. However, Einstein did note that accelerated clocks at different positions run at different rates with respect to the inertial frame.

[6] These coordinates obey the metric,

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 \left(1 + \frac{gz}{c^2}\right)^2 dt^2, \qquad (|z| < c^2/g), \tag{10}$$

where $g = 2\pi G\rho$, G is Newton's gravitational constant and ρ is the density of mass/energy. See, for example, sec. 97 of [8]. This metric is often used to describe a weak gravitational field.

The metric (10) may have been first used by Kottler (1914) [9], and more clearly in sec. VII of [10].

Note that,

$$\sqrt{1-v^2/c^2} = \frac{1}{\cosh gt'/c}\,, \tag{18}$$

The (inertial) lab-frame acceleration of a point at constant z' is,

$$a = \frac{dv}{dt} \;=\; \frac{g}{1+gz'/c^2}\frac{1}{\left[1+\left(\frac{gct}{gz'+c^2}\right)^2\right]^{3/2}} = \frac{g}{1+gz'/c^2}\frac{1}{\cosh^3\frac{gt'}{c}} = \frac{g(1-v^2/c^2)^{3/2}}{1+gz'/c^2}. \tag{19}$$

Recalling eq. (1) we see that the acceleration of point z' in its instantaneous inertial rest frame is,

$$a^\star = \frac{g}{1+gz'/c^2}\,, \tag{20}$$

which depends on the position z' in the accelerated frame. This further emphasizes the difference between a "rigid" accelerated frame and a collection of observers whose acceleration is the same in the lab frame.

The distance between nearby points in the accelerated frame, as measured at a fixed time t in the lab frame, follows from eq. (16),

$$dz = \frac{dz'}{\cosh gt'/c} = \sqrt{1-v^2/c^2}\,dz' \qquad \text{(constant } t\text{)}. \tag{21}$$

Lab-frame observers find that, at time t, lengths in the "rigid" accelerated frame are Lorentz contracted, as expected, according to their instantaneous lab-frame velocity v, when the measurements are made at constant t.

Similarly, observers in the accelerated frame at time t' of a small length dz' find that corresponding length dz in the lab frame is related according to eq. (13) by,

$$dz = dz'\,\cosh\frac{gt'}{c} = \frac{dz'}{\sqrt{1-v^2/c^2}} \qquad \text{(constant } t'\text{)}. \tag{22}$$

That is, the lengths of objects in the lab frame are all also Lorentz contracted, when observed from the "rigid" accelerated frame at constant t'.

The relation between time intervals in the lab and accelerated frames for clocks at fixed z' follows from eq. (14) as,

$$dt(z') = dt'\left(1+\frac{gz'}{c^2}\right)\cosh\frac{gt'}{c} = \frac{dt'(z')}{\sqrt{1-v^2/c^2}}\left(1+\frac{gz'}{c^2}\right) \qquad \text{(constant } z'\text{)}, \tag{23}$$

In particular a clock at $z'=0$ is related by the time dilation,

$$dt_0 = \frac{dt_0'}{\sqrt{1-v^2/c^2}}\,. \tag{24}$$

As all clocks in the inertial lab frame run at the same rate, we can take $dt(z') = dt_0$ to find,

$$dt'(z') = dt'_0 \left(1 + \frac{gz'}{c^2}\right). \tag{25}$$

That is, clocks at larger z' in the accelerated frame run faster than clocks at smaller z', relative to clocks in the inertial lab frame, as noted by Einstein in 1907 [11].

Likewise, using eq. (13) to eliminate z' from eq. (14) in favor of z, we find,

$$t = \frac{gz + c^2}{cg} \tanh \frac{gt'}{c}, \tag{26}$$

and hence,

$$dt = dt' \left(1 + \frac{gz}{c^2}\right) \frac{1}{\cosh^2 gt'/c} = dt' \left(1 - \frac{v^2}{c^2}\right) \left(1 + \frac{gz}{c^2}\right) \qquad \text{(constant } z\text{)}, \tag{27}$$

Clocks at fixed z appear to observers in the accelerated from to run slow (time dilation), but by a factor $1 - v^2/c^2$ rather than $\sqrt{1 - v^2/c^2}$. In addition, this time-dilation factor varies with the coordinate of the clock in the lab frame.

For completeness we note that eqs. (13) and (26) can be combined to give,

$$\frac{1}{\cosh gt'/c} = \sqrt{1 - \left(\frac{gct}{gz + c^2}\right)^2}, \tag{28}$$

$$z' = \frac{z + c^2/g}{\cosh gt'/c} - \frac{c^2}{g} = \frac{gz + c^2}{g} \sqrt{1 - \left(\frac{gct}{gz + c^2}\right)^2} - \frac{c^2}{g}, \tag{29}$$

The inverses of transformations of (11)-(14) are,

$$x' = x, \tag{30}$$

$$y' = y, \tag{31}$$

$$z' = \sqrt{\left(z + \frac{c^2}{g}\right)^2 - c^2 t^2} - \frac{c^2}{g}, \tag{32}$$

$$ct' = \frac{c^2}{g} \tanh^{-1} \left(\frac{ct}{z + c^2/g}\right). \tag{33}$$

B Appendix: Bell's Spaceship Paradox

An interesting example of the difference between a "rigid" accelerated frame and a collection of observers with the same lab-frame accelerations was given by Dewan and Beran [12, 13], and popularized by Bell [14].

Here, two spaceships move, with a rope connecting them, along the z-axis with identical accelerations and constant separation dz for any time t in the inertial lab frame. Then, according to eq. (21), the separation of the spaceships in the accelerated frame of, say, the

left spaceship is $dz' = dz/\sqrt{1 - v^2/c^2} > dz$. In the frame of either of the spaceships the rope appears to be stretched, and eventually breaks.

This result is very disconcerting to those who think that the spaceships define a "rigid" accelerated frame, in which the distance between two points would be independent of time. But, as discussed around eq. (22), the distance between the spaceships is increasing in the "rigid" accelerated frame associated with either of the spaceships, so it should be no surprise that the rope eventually breaks.

According to the Equivalence Principle, a uniformly accelerated frame is equivalent to a frame at rest in a uniform gravitational field. An object at rest in a uniform gravitational field has a constant length, as does an object in a uniformly accelerated frame (according to observers in that frame). However, many people seem to suppose that the two spaceships in Bell's paradox define a uniformly accelerated frame, in which case the rope should not be expected to break. Or, if one accepts that the rope breaks, but one supposes that the two spaceships define a uniformly accelerated frame, then according to the Equivalence Principle, a rope suspended at rest in a uniform gravitational field would be expected to break after a while.

These paradoxes reinforce the insight of Appendix A that a uniformly accelerated frame is not a collection of observers with the same acceleration in the inertial lab frame.[7]

C Appendix: Additional Remarks

C.1 Comments by Einstein on the Equivalence Principle

In 1920, Einstein [43] recalled his invention of the Equivalence Principle: *When I was busy (in 1907) writing a summary of my work on the theory of special relativity for the Jahrbuch für Radioaktivität und Elektronik [11], I also had to try to modify the Newtonian theory of gravitation such as to fit its laws into the theory. While attempts in this direction showed the practicability of this enterprise, they did not satisfy me because they would have had to be based upon unfounded physical hypotheses. At that moment I got the happiest thought of my life in the following form:*

In an example worth considering, the gravitational field has a relative existence only in a manner similar to the electric field generated by magneto-electric induction. Because for an observer in free-fall from the roof of a house there is during the fall – at least in his immediate vicinity – no gravitational field. Namely, if the observer lets go of any bodies, they remain relative to him, in a state of rest or uniform motion, independent of their special chemical or physical nature. The observer, therefore, is justified in interpreting his state as being "at rest".

Einstein's first published statement of the Equivalence Principle was at the end of sec. 17 of [11] (1907): *we ... assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.*[8]

[7] Additional commentaries on this theme, of possible amusement, are in [15]-[37]. A related debate on the acceleration of extended objects can be traced in [38]-[42].

[8] Some people, perhaps including Einstein (see, for example, [46]), interpret the "complete physical equivalence" to mean that there could be no physical system without a gravitational field. Others consider

Einstein's second published statement of the Equivalence Principle (1911) was at the end of sec.1 of [44]: *By theoretical consideration of processes which take place relative to a system of reference with uniform acceleration, we obtain information as to the behavior of processes in a homogeneous gravitational field.*

Einstein's comments on the Equivalence Principle in sec. 2 of his 1916 review of general relativity [45] include: *Let K' be a second system of reference which is moving relative to K in* uniformly accelerated *translation. Then relative to K', a mass sufficiently distant from other masses would have an accelerated motion such that its acceleration and direction of acceleration are independent of the material composition and physical state of the mass.*

Does this permit an observer at rest relative to K' to infer that he is on a "really" accelerated system of reference? The answer is in the negative; for the above-mentioned of freely movable masses relative to K' may equally well be interpreted in the following way. The system of reference K' is unaccelerated, but the space-time territory in question is under the sway to a gravitational field which generates the accelerated motion of the bodies relatively to K'.

This view is made possible for us by the teaching of experience of the existence of a field of force, namely the gravitational field, which possesses the remarkable property of imparting the same acceleration to all bodies.

Here, Einstein supposes that the acceleration due to gravity of a mass is independent of the velocity of the mass, even though he had shown that in his theory the gravitational deflection of light is twice the "Newtonian" value (sec. 22 of [45]).

C.2 Limitations to the Equivalence Principle

The equivalence applies only locally, not globally, as nonuniform gravitational fields exist.[9] This was noted briefly in Einstein's above comments of 1920 [43].

A feature which is seldom mentioned is that the version of the Equivalence Principle which states that the acceleration due to gravity is independent of the "physical state of the mass" applies only to the behavior of objects with speeds much less than that of light. See, for example, [47], which reviews how the acceleration of an object due to gravity depends on the velocity of the object, and how the force of gravity for radial motion with $v_r > c/\sqrt{3}$, with respect to a spherical source, is repulsive rather than attractive.

C.3 Some Consequences of the Equivalence Principle

A consequence of the Equivalence Principle is that an observer at rest with respect to the source of a gravitational field is equivalent to an accelerated observer in zero gravity.[10]

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that there could be spacetime without a gravitational field, namely "flat" spacetime in which there is no curvature (zero Ricci scalar = zero gravity). The comments below reflect the latter attitude.

[9] Nonuniform gravitational fields are often said to generate "tidal forces", which vary with position in space. The Equivalence Principle for a nonuniform gravitational field holds only to the extent that these "tidal forces" are negligible to the local observer.

[10] As noted above, we take "zero gravity" to mean "zero Ricci scalar", which includes the idealization of a completely uniform "gravitational" field. In this view, the Equivalence Principle is somewhat trivial when applied to a uniform "gravitational" field, which is simply an interpretation of flat spacetime by a uniformly accelerated observer.

Einstein's "happy thought" of 1907 was that an observer in free fall in gravity (who is accelerated with respect to an observer at rest in the gravity) is equivalent to an inertial observer in zero gravity.[11]

(*The next two paragraphs summarize results from sec. 2 above.*) Note also that if the acceleration of gravity is g at some point at rest in a gravitational field, then the equivalent acceleration in zero gravity is g according to an accelerated observer in an inertial frame, rather than according to an inertial observer. Since the g of gravity could be static, and last forever, it cannot correspond to a constant acceleration g with respect to an inertial frame in zero gravity, which would imply eventual speeds faster than light.

That is, constant acceleration with respect to an inertial frame in zero gravity is not equivalent to a constant/uniform gravitational field.[12]

Observers at rest in an inertial frame maintain a constant distance between one another.

Observers in free fall in uniform gravity keep a constant distance between one another, if they start falling from rest at the same moment/time.

Observers in zero gravity who start from rest at the same time in some inertial frame, and are accelerated uniformly, maintain a constant separation according to observers in the original inertial frame. However, according to the accelerated observers, their separation increases with time. This is Bell's spaceship paradox, discussed in Appendix B above. Furthermore, the distance between two objects at rest in the inertial frame (along a line parallel to the acceleration vector) appears to decrease with time according to the accelerated observer, as discussed in Appendix A above.

C.4 Two Objects Accelerated with Respect to an Inertial Frame

Two objects can accelerate with respect to an inertial frame in zero gravity, with constant $a = g$ in their accelerated frames, such that the distance between the two objects remains constant with respect to the inertial frame. However, the distance between these two objects is not constant with respect to their accelerated frames (which are different for the two objects).

The equivalent of this is two objects that remain at rest, with constant separation, in a constant/uniform gravitational field of g.

C.5 Two Objects at Rest in an Inertial Frame

The equivalent of objects in an inertial frame in zero gravity is free-falling objects in gravity.

The equivalent of the two objects in an inertial frame in zero gravity having constant separation is that they free fall in gravity such that their separation is constant in their own free-falling frames. This does not mean that their separation is constant with respect to observers at rest in gravity (which are the equivalents of accelerated observers in zero

[11] Not all observers who are accelerated with respect to observers who are at rest in gravity are equivalent to inertial observers in zero gravity. Only free-falling observers in gravity are equivalent to inertial observers in zero gravity.

[12] Constant acceleration with respect to an inertial frame in zero gravity is not equivalent to a constant/uniform gravitational field.

gravity). Recall from Appendix A above that according to an accelerated observer in zero gravity of two objects at rest in an inertial frame, their separation decreases as their velocity relative to the accelerated observer increases in magnitude. Hence, the equivalent is that the separation of the free falling objects decreases as they fall, according to observers at rest in the gravity. This mean that the "upper" object started free falling earlier than the "lower" object.

D Appendix: Uniform Gravitational Field[13]

The notion of a uniform gravitational field is somewhat elusive. If one associates gravitational fields with sources of mass/energy, then physical gravitational fields are typically associated with distortions of spacetime.[14] On the other hand, the equivalence principle implies that a uniformly accelerated reference frame in flat spacetime should be equivalent to a uniform gravitational field. Of course, a uniform field over all spacetime is a mathematical idealization, such that there is room for discussion as to the relevant physical approximation to this concept. Lengthy debate on this topic may or may not have converged, but present wisdom seems to be that reasonably physical assumptions as to the sources of a uniform gravitational field are consistent with it being associated with flat spacetime [11],[54]-[65].

Often a weak, uniform gravitational is taken to be described by the metric,[15]

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\left(1 + \frac{gz}{c^2}\right)^2 dt^2, \qquad (|z| < c^2/g), \tag{34}$$

where $g = 2\pi G\rho$, G is Newton's gravitational constant and ρ is the density of mass/energy. See, for example, sec. 97 of [8].

For spacetime described by the static metric (34), electrodynamics obey Maxwell's equation with the alterations that the vacuum has relative permittivity and permeability given by,

$$\epsilon = \mu = \frac{1}{1 + gz/c^2}\,, \tag{35}$$

as discussed, for example, in sec. 90 of [48]. A consequence is that the speed, u, of light

[13]This Appendix is transcribed from secs. secs. 2.2-3 of [66].

[14]These distortions are often called "curvature", but in the case of hypothetical "cosmic strings" and "domain walls" [50, 51] spacetime is flat with topological defects. Vacuum "domain walls" are not physically viable, but remain an interesting theoretical construct.

[15]The metric (34) may have been first used by Kottler (1914) [9], and more clearly in sec. VII of [10].

emitted at $z = 0$ is a function of z according to,[16,17]

$$u(z) = c\left(1 + \frac{gz}{c^2}\right). \tag{36}$$

If we approximate a uniform gravitational field by that at the surface of the Earth, then the symbol g in eq. (36) becomes, approximately, $g_0(1 - z^2/2R_E^2)$ where $g_0 = GM_E/R_E^2$, G is Newton's gravitational constant, M_E and R_E are the mass and radius of the Earth, respectively.

D.1 Does a Uniform Gravitational Field Have a Source?

Using coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, the metric tensors g_{ij} and g^{ij} corresponding to eq. (34) have nonzero components,[18]

$$g_{00} = \frac{1}{g^{00}} = f^2(z) = \left(1 + \frac{gz}{c^2}\right)^2, \qquad g_{11} = g_{22} = g_{33} = g^{11} = g^{22} = g^{33} = -1, \tag{37}$$

such that $g_{ik}\, g^{jk} = \delta_i^j$. The nonzero Christoffel symbols are,

$$\Gamma_{i,jk} = \Gamma_{i,kj} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \qquad \Gamma_{0,03} = \Gamma_{0,30} = -\Gamma_{3,00} = f\frac{df}{dz} \equiv ff'. \tag{38}$$

The Riemann curvature tensor has nonzero components,

$$R_{ijkl} = \frac{\partial \Gamma_{i,jl}}{\partial x^k} - \frac{\partial \Gamma_{i,jk}}{\partial x^l} + g^{mn}\Gamma_{i,mk}\Gamma_{n,jl} - g^{mn}\Gamma_{i,ml}\Gamma_{n,jk}, \tag{39}$$

$$R_{0330} = R_{3003} = -R_{0303} = -R_{3030} = ff''. \tag{40}$$

[16]Equation (36) appears near the end of Einstein's 1907 paper [11].

[17]Our brief discussion avoids the issue of variation with z of the rate of clocks in a uniform gravitational field. However, the metric (34) indicates that a clock (that reads time t) at position z has proper time interval $d\tau = (1 + gz/c^2)dt$, such clock at $z > 0$ runs slower compared to proper time than a clock at $z = 0$. Hence, reporting the speed of light at position $z > 0$ as $u(z) = dz/dt = (dz/d\tau)(d\tau/dt) = c\,(1 + gz/c^2)$ gives a value larger than c. If light is emitted in the $+z$-direction at $z = -c^2/g$ its initial speed is zero according to eq. (36), such that it takes an infinite time interval Δt to reach $z = 0$, and we speak of $z = -c^2/g$ as the "event horizon" for the observer at $z = 0$. However, an observer at $z = -c^2/g$ could consider that the light has local speed c, and the metric to be eq. (34) with z replaced by $z + c^2/g$, such that the speed of light varies with z according to $u(z) = c\,(1 + g(z + c^2/g)/c^2) = c\,(2 + gz/c^2)$, and the event horizon for this observer is $z = -2c^2/g$. Similarly, an observer at $z = c^2/g$ who considers the local speed of light to be c concludes that light emitted at $z = 0$ takes an infinite time to reach him, so that in effect an observer at $z = 0$ cannot communicate with one at $z = c^2/g$. Hence, we say that the metric (34) is valid only for $|z| < c^2/g$.

Another way to see this is to note that the gravitational redshift brings the energy of any photon emitted at $z = 0$ to zero at $z = c^2/g$ [52], so there is no meaningful physical interaction possible between an observer at $z = 0$ and one at $z > c^2/g$.

A universe with a uniform gravitational field is effectively partitioned into regions of extent $\Delta z = \pm c^2/g$ around any observer. Each observer cannot know about the rest of the universe outside this domain. That is, early cosmological visions that assumed a flat Earth and "turtles all the way down" were actually consistent with general relativity.

[18]For the general case of symmetric metric tensors, see prob. 2, sec. 92 of [48].

The Ricci tensor has nonzero components,

$$R_{ij} = g^{kl} R_{kilj}, \qquad R_{00} = f f'', \qquad R_{33} = -\frac{f''}{f}\,. \tag{41}$$

The Ricci curvature scalar is,[19]

$$R = g^{ij} R_{ij} = \frac{2f''}{f}\,. \tag{42}$$

Einstein's gravitational equations are,

$$\frac{8\pi G}{c^4} T_{ij} = R_{ij} - g_{ij} R, \tag{43}$$

$$T_{00} = -\frac{c^4}{8\pi G} f f'', \qquad T_{11} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{22} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{33} = \frac{c^4}{8\pi G}\frac{f''}{f}\,. \tag{44}$$

Hence, the choice $f(z) = 1 + gz/c^2$, for which $f'' = 0$, implies that the stress-energy tensor T_{ij} is everywhere zero. The "uniform gravitational field" corresponding to the metric (37) has no source, or spacetime curvature, and is only a kind of "coordinate force" akin to the centrifugal force and the Coriolis force.[20,21]

Requiring a uniform gravitational field to have an infinite planar source and flat spacetime apparently leads to metrics with spatial anisotropy. See, for example, [50, 54],[60]-[65].

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[19] Probably, $R = f''/f$, such that $T_{00} = T_{33} = 0$, and I have errors somewhere.

[20] The metric (37) is valid only for $z > z_0 = -c^2/g$, which leaves open the possibility of sources at $z < z_0$, and in particular a plane sheet of mass at $z = z_0$.

[21] As the nonphysical, mathematical idealization of a "uniform gravitational field" is associated with flat spacetime (zero Ricci scalar), many people (including this author) consider it not to be an actual gravitational field. However, others consider that even flat spacetime is a kind of gravitational field.

If one supposes that there can exist a gravitational field associated with flat spacetime, that field can have arbitrary strength. This leads to the "hole problem" that delayed progress by Einstein in 1913-1915 that more that one gravitational field might be associated with any particular set of masses. See, for example, [67]-[69].

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