Symmetries of Electromagnetic Fields Associated with a Plane Conducting Screen

Zhong Ming Tan and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(January 14, 2012; updated August 31, 2012)

1 Problem

Consider a set of charge and current sources located at, say, $z < 0$ whose electromagnetic fields are modified by a conducting screen, with apertures, on the plane $z = 0$, and there are no other charges or currents anywhere. The total electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be decomposed at the sum of "incident" and "scattered" fields,

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s, \qquad \mathbf{B} = \mathbf{B}^i + \mathbf{B}^s, \tag{1}$$

where the incident fields are those associated with the sources at $z < 0$ in the absence of the screen, and the scattered field are those due only to the charges and currents on the screen.

In, for example, sec. 11.2 of [1] it is claimed that the scattered fields obey the symmetries,

$$\mathbf{E}^s_x(x, y, -z) = \mathbf{E}^s_x(x, y, z), \qquad \mathbf{B}^s_x(x, y, -z) = -\mathbf{B}^s_x(x, y, z), \tag{2}$$

$$\mathbf{E}^s_y(x, y, -z) = \mathbf{E}^s_y(x, y, z), \qquad \mathbf{B}^s_y(x, y, -z) = -\mathbf{B}^s_y(x, y, z), \tag{3}$$

$$\mathbf{E}^s_z(x, y, -z) = -\mathbf{E}^s_z(x, y, z), \qquad \mathbf{B}^s_z(x, y, -z) = \mathbf{B}^s_z(x, y, z). \tag{4}$$

Explain why these (and other) symmetries hold.

A significant application of the symmetries (2)-(4) is in the justification of the electromagnetic version of Babinet's principle of complementary screens. See, for example, [2].

2 Solution

2.1 The Symmetries Hold Because No Currents Cross the Edge

An argument as to why the symmetries (2)-(4) hold in general can be based on the discussion in sec. 10.7 of [3] (which did not appear in earlier editions of that work). Namely, the scattered fields can be deduced from scalar potential V^s and vector potential $\mathbf{A}^s$ due to the charges and currents on the screen, which potentials are even functions of z. Then, (in Gaussian units, with c being the speed of light in vacuum),

$$\mathbf{E}^s = -\boldsymbol{\nabla} V^s - \frac{1}{c}\frac{\partial \mathbf{A}^s}{\partial t}, \qquad \mathbf{B}^s = \boldsymbol{\nabla} \times \mathbf{A}^s, \tag{5}$$

and,

$$E^s_z = -\frac{\partial V^s}{\partial z} - \frac{1}{c}\frac{\partial A^s_z}{\partial t}, \qquad B^s_x = \frac{\partial A^s_z}{\partial y} - \frac{\partial A^s_y}{\partial z}, \qquad \text{and} \qquad B^s_y = \frac{\partial A^s_x}{\partial z} - \frac{\partial A^s_z}{\partial x}, \tag{6}$$

have no symmetry in z if A_z^s is nonzero, while E_x^s, E_y^s and B_z^s do obey the symmetries (2)-(3).

In principle, currents could flow from one side of the screen to the other, such that the current density J_z^s is nonzero on the edge(s) of the screen, leading to nonzero A_z^s. However, as first noted by Meixner [4], such current density would give rise to a magnetic field whose stored energy diverges logarithmically within finite volumes surrounding portions of the edge.[1,2] The physical requirement that the scattered field energy (which derives from the incident wave) be finite inside finite volumes implies that J_z^s, and also A_z^s, must be zero. Hence, the symmetries (2)-(4) do hold in general.

These forms seem to have first appeared in [5] (based on lemma 2, p. 508), but they were not well justified in that work. An argument similar to the above for these forms is given in sec. 9.2 of [6]. A review of edge conditions for plane conducting screens is given in [7].

Experiments in which polarization effects in scattering of light from a knife edge were first performed by Gouy in 1883 [8]. A partial explanation was given by Poincaré in 1892 [9], which is perhaps the first calculation of diffraction of electromagnetic waves. This first "complete" solution for an electromagnetic diffraction problem was by Sommerfeld in 1895 [10, 11, 12, 13], who found scattered fields that obey the symmetries (2)-(4), although this is perhaps not obvious. Other early examples of the scattering of electromagnetic waves by plane conducting screens were considered by Rayleigh [14, 15], for "small" apertures and waves with normal incidence, and by Lamb [16] gave an "exact" analysis of waves normally incident on a screen with period slots of any size, who found fields that obey the symmetries (2)-(4), as reviewed in the Appendix below.

2.2 Incident, Reflected and Transmitted Fields

A different decomposition of the fields than eq. (1) can be made in terms of incident, reflected and transmitted fields,

$$\mathbf{E}(z<0) = \mathbf{E}^i + \mathbf{E}^r, \qquad \mathbf{E}(z>0) = \mathbf{E}^t, \qquad \mathbf{B}(z<0) = \mathbf{B}^i + \mathbf{B}^r, \qquad \mathbf{B}(z>0) = \mathbf{B}^t. \quad (7)$$

Lamb [16] made the further decomposition,

$$\mathbf{E}^r = \mathbf{E}^{r_0} + \mathbf{E}^{r_1}, \qquad \mathbf{B}^r = \mathbf{B}^{r_0} + \mathbf{B}^{r_1}, \tag{8}$$

where $\mathbf{E}^{r_0}$ and $\mathbf{B}^{r_0}$ are the reflected fields (for $z<0$) if the entire plane $z=0$ were a perfect conductor.[3]

[1]For example, if the conducting half plane $(x<0,y,0)$ had an edge current $I\,\hat{\mathbf{z}}$ per unit length, then the magnetic field would be $B_y = -2Ix/c(x^2+z^2)$, and the field energy in a box with one corner at the origin and diagonal to $(x \ll a, a, a)$ would be $\propto \int B_y^2\, d\mathrm{Vol} \propto (aI^2/c^2)\int_0^x x^2\,dx \int_0^a dz/(x^2+z^2)^2 = K + (\pi a I^2/c^2)\int_0^x dx/x$, which diverges logarithmically. If the edge current existed only over the line segment $(0, -b<y<b, 0)$, then the above calculation would still be valid for a box with $a \ll b$.

[2]The surface charge density near the edge varies inversely with the square root of the distance from the edge, such that the electric field energy remains finite in finite volumes surrounding the edge.

[3]The corresponding decomposition of the transmitted field,

$$\mathbf{E}^t = \mathbf{E}^{t_0} + \mathbf{E}^{t_1}, \qquad \mathbf{B}^t = \mathbf{B}^{t_0} + \mathbf{B}^{t_1}, \tag{9}$$

is trivial in that $\mathbf{E}^{t_0} = 0 = \mathbf{B}^{t_0}$.

The reflected fields in case the entire plane $z = 0$ is a perfect conductor can be calculated from the image charge and current densities at $z > 0$,

$$\rho^{\rm image}(x, y, z) = -\rho(x, y, -z), \tag{10}$$

$$\begin{aligned} J_x^{\rm image}(x, y, z) &= -J_x(x, y, -z), \\ J_y^{\rm image}(x, y, z) &= -J_y(x, y, -z), \\ J_z^{\rm image}(x, y, z) &= J_z(x, y, -z). \end{aligned} \tag{11}$$

A general prescription for computing electromagnetic fields is given by so-called vector diffraction theory. One formulation of this has been given by Stratton and Chu [17, 18],[4] assuming all source charge and current densities ρ and $\mathbf{J}$ and fields $\mathbf{E}$ and $\mathbf{B}$ have time dependence $e^{-i\omega t}$,

$$\mathbf{E}(\mathbf{x}) = \int_V \left(\frac{ik}{c} \mathbf{J}(\mathbf{x}') \frac{e^{ikr}}{r} + \rho(\mathbf{x}') \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \right) d{\rm Vol}' \tag{12}$$

$$-\frac{1}{4\pi} \oint_S \left\{ ik[\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \frac{e^{ikr}}{r} + [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \times \boldsymbol{\nabla}' \frac{e^{ikr}}{r} + [\hat{\mathbf{n}}' \cdot \mathbf{E}(\mathbf{x}')] \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \right\} d{\rm Area}',$$

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c} \int_V \mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \, d{\rm Vol}' \tag{13}$$

$$+\frac{1}{4\pi} \oint_S \left\{ ik[\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \frac{e^{ikr}}{r} - [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \times \boldsymbol{\nabla}' \frac{e^{ikr}}{r} - [\hat{\mathbf{n}}' \cdot \mathbf{B}(\mathbf{x}')] \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \right\} d{\rm Area}',$$

where Gaussian units are employed, $\hat{\mathbf{n}}'$ is the outward unit vector normal to surface S (that bounds volume V), and $r = |\mathbf{x} - \mathbf{x}'|$.

We first suppose the screen is absent, and volume V to be all of space. The surface S is at "infinity", and we assume that in physically realistic examples the surface integrals at "infinity" vanish. Then, the "incident" fields can be computed by,

$$\mathbf{E}^i(\mathbf{x}) = \int_{z'<0} \left(\frac{ik}{c} \mathbf{J}(\mathbf{x}') \frac{e^{ikr}}{r} + \rho(\mathbf{x}') \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \right) d{\rm Vol}' \qquad \text{(no screen)}, \tag{14}$$

$$\mathbf{B}^i(\mathbf{x}) = \frac{1}{c} \int_{z'<0} \mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \, d{\rm Vol}' \qquad \text{(no screen)}, \tag{15}$$

assuming that the sources are in the region $z < 0$. Similarly, the reflected fields when all of $z = 0$ is a perfect conductor are given by,

$$\mathbf{E}^{r_0}(\mathbf{x}) = \int_{z'>0} \left(\frac{ik}{c} \mathbf{J}^{\rm image}(\mathbf{x}') \frac{e^{ikr}}{r} + \rho^{\rm image}(\mathbf{x}') \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \right) d{\rm Vol}', \tag{16}$$

$$\mathbf{B}^{r_0}(\mathbf{x}) = \frac{1}{c} \int_{z'>0} \mathbf{J}^{\rm image}(\mathbf{x}') \times \boldsymbol{\nabla}' \frac{e^{ikr}}{r} \, d{\rm Vol}'. \tag{17}$$

Then, recalling eq. (11), the incident and reflected fields (in case the plane $z = 0$ is a perfect conductor) obey the relations (where z is positive),

$$\mathbf{E}_x^{r_0}(x, y, -z) = -\mathbf{E}_x^i(x, y, z), \qquad \mathbf{B}_x^{r_0}(x, y, -z) = \mathbf{B}_x^i(x, y, z), \tag{18}$$

[4] See also the Appendix of [19].

$$\mathbf{E}_y^{r_0}(x,y,-z) = -\mathbf{E}_y^i(x,y,z), \qquad \mathbf{B}_y^{r_0}(x,y,-z) = \mathbf{B}_y^i(x,y,z), \tag{19}$$

$$\mathbf{E}_z^{r_0}(x,y,-z) = \mathbf{E}_z^i(x,y,z), \qquad \mathbf{B}_z^{r_0}(x,y,-z) = -\mathbf{B}_z^i(x,y,z). \tag{20}$$

In general, the reflected fields are the same as the scattered fields for $z < 0$, so we can write,

$$\mathbf{E}^{r_1} = \mathbf{E}^r - \mathbf{E}^{r_0} = \mathbf{E}^s(z<0) - \mathbf{E}^{r_0}, \tag{21}$$

$$\mathbf{B}^{r_1} = \mathbf{B}^r - \mathbf{B}^{r_0} = \mathbf{B}^s(z<0) - \mathbf{B}^{r_0}. \tag{22}$$

The transmitted fields (for $z > 0$) follow from eqs. (5) as,

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s(z>0), \tag{23}$$

$$\mathbf{B}^t = \mathbf{B}^i + \mathbf{B}^s(z>0). \tag{24}$$

Recalling the symmetries (2)-(4) and (18)-(20), eqs. (21)-(24) lead to the additional symmetries,

$$\mathbf{E}_x^{r_1}(x,y,-z) = \mathbf{E}_x^t(x,y,z), \qquad \mathbf{B}_x^{r_1}(x,y,-z) = -\mathbf{B}_x^t(x,y,z), \tag{25}$$

$$\mathbf{E}_y^{r_1}(x,y,-z) = \mathbf{E}_y^t(x,y,z), \qquad \mathbf{B}_y^{r_1}(x,y,-z) = -\mathbf{B}_y^t(x,y,z), \tag{26}$$

$$\mathbf{E}_z^{r_1}(x,y,-z) = -\mathbf{E}_z^t(x,y,z), \qquad \mathbf{B}_z^{r_1}(x,y,-z) = \mathbf{B}_z^t(x,y,z), \tag{27}$$

where $z > 0$.

The symmetries (25)-(27) have been claimed to be evident in [12, 20, 21].[5]

Appendix: Lamb's Example

An early "exact" analysis of electromagnetic fields associated with a plane conducting screen appeared in the 1898 paper by Lamb [16] (secs. 5 and 6),[6] but seems to be little known. Here, we transcribe the discussion of Lamb's example given in sec. 2.2 of [24] into the present notation, to show that the fields in this example satisfy both the symmetries (2)-(4) and (25)-(27).[7]

The plane $z = 0$ is a perfect conductor except for slots with period d running along the x direction. One of the slots is centered on the line $(x, 0, 0)$. The remaining strips of perfect conductor have width $2a$, and we define $1/\mu = \sin(\pi a/d)$. Lamb made a deft use of conjugate functions to identify a useful function,

$$u(y,z) = \frac{\pi\,|z|}{d} - \ln\frac{1}{\mu} + \sum_{n=1}^{\infty} C_n\, e^{-2n\pi|z|/d} \cos\frac{2n\pi y}{d}\,. \tag{28}$$

[5] Although [21] is largely a transcription of Schwinger's famous wartime paper [22], this claim does not appear in the latter.

[6] Lamb built on earlier efforts of Thomson [23] and Rayleigh [14, 15].

[7] Lamb never mentioned charges or currents in his discussion, but his analysis is of a mathematical "monolayer" with the tacit assumption that no currents flow in the z-direction. Hence, we expect the symmetries (2)-(4) and (25)-(27) to hold, following footnotes 3 and 7.

Then, a plane wave normally incident on this screen from $z < 0$ with electric field polarized in the x-direction (parallel to the strips) results in total electric field given by,

$$
\begin{aligned}
E_x(z<0) &= E_0\, e^{i(kz-\omega t)} + E_1\, e^{-i(kz+\omega t)} + E_2 \left(u + \frac{\pi z}{d} + \ln \frac{1}{\mu} \right) \\
&= E_x^i + E_x^s(z<0) = E_x^i + E_x^r = E_x^i + E_x^{r_0} + E_x^{r_1}, \qquad (29) \\
E_x(z>0) &= (E_0 + E_1)\, e^{i(kz-\omega t)} + E_2 \left(u - \frac{\pi z}{d} + \ln \frac{1}{\mu} \right) \\
&= E_x^i + E_x^s(z>0) = E_x^t, \qquad (30)
\end{aligned}
$$

where,

$$
\frac{E_1}{E_0} = -\frac{1}{1 + iCkd}, \qquad E_2 = \frac{ikdE_1}{\pi}, \qquad \text{with} \qquad C = \frac{1}{\pi} \ln \frac{1}{\mu} = \frac{1}{\pi} \ln \sin \frac{\pi a}{d}. \qquad (31)
$$

The reflected wave in case the plane $z = 0$ were a perfect conductor is $E_x^{r_0} = -E_0\, e^{-i(kz+\omega t)}$, so that,

$$
E_x^{r_1}(z<0) = (E_0 + E_1)\, e^{i(kz-\omega t)} + E_2 \left(u + \frac{\pi z}{d} + \ln \frac{1}{\mu} \right), \qquad (32)
$$

and eqs. (30)-(32) obey symmetry (2), $E_x^{r_1}(x, y, -z) = E_x^t(x, y, z)$, as well as the symmetry (25), $E_x^s(x, y, -z) = E_x^s(x, y, z)$.

Likewise, a plane wave normally incident on this screen from $z < 0$ with magnetic field polarized in the x-direction (such that the incident electric field is perpendicular to the strips) results in total magnetic field given by,

$$
\begin{aligned}
B_x(z<0) &= E_0\, e^{i(kz-\omega t)} + E_1\, e^{-i(kz+\omega t)} + E_2 \left(u + \frac{\pi z}{d} + \ln \frac{1}{\mu} \right) \\
&= B_x^i + B_x^s(z<0) = B_x^i + B_x^r = E_x^i + E_x^{r_0} + E_x^{r_1}, \qquad (33) \\
B_x(z>0) &= (E_0 - E_1)\, e^{i(kz-\omega t)} - E_2 \left(u - \frac{\pi z}{d} + \ln \frac{1}{\mu} \right) \\
&= B_x^i + B_x^s(z>0) = B_x^t, \qquad (34)
\end{aligned}
$$

where in this case the slots are located where the conducting strips were previously. Here, the slots have width $2a$, the function u is still given by eq. (28) but with $1/\mu = \cos(\pi a/d)$,

$$
\frac{E_1}{E_0} = \frac{iDkd}{1 + iDkd}, \qquad E_2 = -\frac{E_1}{\ln \frac{1}{\mu}}, \qquad \text{with} \qquad D = \frac{1}{\pi} \ln \frac{1}{\mu} = \frac{1}{\pi} \ln \cos \frac{\pi a}{d}. \qquad (35)
$$

The reflected wave in case the plane $z = 0$ were a perfect conductor is $B_x^{r_0} = E_0\, e^{-i(kz+\omega t)}$, so that,

$$
B_x^{r_1}(z<0) = (E_0 - E_1)\, e^{i(kz-\omega t)} + E_2 \left(u + \frac{\pi z}{d} + \ln \frac{1}{\mu} \right), \qquad (36)
$$

and eqs. (34)-(36) obey symmetry (2), $B_x^{r_1}(x, y, -z) = -B_x^t(x, y, z)$, as well as the symmetry (25), $B_x^s(x, y, -z) = -B_x^s(x, y, z)$.

References

[1] M. Born and E. Wolf, *Principles of Optics*, 7$^{\rm th}$ ed. (Cambridge U. Press, 1999), http://kirkmcd.princeton.edu/examples/EM/born_wolf_7ed.pdf

[2] Z.M. Tan and K.T. McDonald, *Babinet's Principle for Electromagnetic Fields* (Jan. 19, 2012), http://kirkmcd.princeton.edu/examples/babinet.pdf

[3] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[4] J. Meixner, *Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Wellen an vollkommen leitenden ebenen Schirmen*, Ann. Phys. **441**, 1 (1950), http://kirkmcd.princeton.edu/examples/EM/meixner_ap_441_1_50.pdf

[5] J. Meixner, *Streng Theorie der Beugung elektromagnetisher Wellen an der vollkommen leitenden Kreisscheibe*, Z. Naturf. **3a**, 506 (1948), http://kirkmcd.princeton.edu/examples/EM/meixner_zn_3a_506_48.pdf

[6] D.S. Jones, *The Theory of Electromagnetism* (Macmillan, 1964), http://kirkmcd.princeton.edu/examples/EM/jones_sec9-2.pdf

[7] A.E. Heins and S. Silver, *The Edge Conditions and Field Representation Theorems in the Theory of Eelectromagnetic Diffraction*, Proc. Camb. Phil. Soc. **51**, 149 (1955), http://kirkmcd.princeton.edu/examples/EM/heins_pcps_51_149_55.pdf

[8] M. Gouy, *Sur la polarisation de la lumière diffractée*, Compt. Rend. Acad. Sci. **96**, 697 (1883), http://kirkmcd.princeton.edu/examples/EM/gouy_cras_96_697_83.pdf

[9] H. Poincaré, *Sur la Polarisation par Diffraction*, Acta Math. **16**, 297 (1892), http://kirkmcd.princeton.edu/examples/EM/poincare_am_16_297_92.pdf

[10] A. Sommerfeld, *Mathematische Theorie der Diffraction*, Math. Ann. **47**, 317 (1896), http://kirkmcd.princeton.edu/examples/EM/sommerfeld_ma_47_317_96.pdf

[11] A. Sommerfeld, *Optics* (German edition 1950, English translation: Academic Press, 1964), http://kirkmcd.princeton.edu/examples/EM/sommerfeld_optics_38-39.pdf

[12] H. Lamb, *On Sommerfeld's Diffraction Problem; and on Reflection by a Parabolic Mirror*, Proc. London Math. Soc. **4**, 190 (1907), http://kirkmcd.princeton.edu/examples/EM/lamb_plms_4_190_07.pdf

[13] K.T. McDonald, *Sommerfeld's Diffraction Problem* (Jan. 22, 2012), http://kirkmcd.princeton.edu/examples/sommerfeld.pdf

[14] Lord Rayleigh, *On the Passage of Waves through Apertures in Plane Screens, and Allied Problems*, Phil. Mag. **43**, 259 (1897), http://kirkmcd.princeton.edu/examples/EM/rayleigh_pm_43_259_97.pdf

[15] Lord Rayleigh, *On the Incidence of Aerial and Electric Waves upon Small Obstacles in the form of Ellipsoids or Elliptic Cylinders, and on the Passage of Electric Waves through a circular Aperture in a Conducting Screen*, Phil. Mag. **44**, 28 (1897), http://kirkmcd.princeton.edu/examples/EM/rayleigh_pm_44_28_97.pdf

[16] H. Lamb, *On the Reflection and Transmission of Electric Waves by a Metallic Grating*, Proc. London Math. Soc. **29**, 523 (1898), http://kirkmcd.princeton.edu/examples/EM/lamb_plms_29_523_98.pdf

[17] J.A. Stratton and L.J. Chu, *Diffraction Theory of Electromagnetic Waves*, Phys. Rev. **56**, 99 (1939), http://kirkmcd.princeton.edu/examples/EM/stratton_pr_56_99_39.pdf

[18] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941), http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

[19] K.T. McDonald, *Time-Reversed Diffraction* (Sept. 5, 1999), http://kirkmcd.princeton.edu/examples/laserfocus.pdf

[20] H. Bethe, *Theory of Diffraction by Small Holes*, Phys. Rev. **66**, 163 (1944), http://kirkmcd.princeton.edu/examples/EM/bethe_pr_66_163_44.pdf

[21] H. Levine and J. Schwinger, *On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen*, Comm. Pure Appl. Math. **3**, 355 (1950), http://kirkmcd.princeton.edu/examples/EM/levine_cpam_3_355_50.pdf

[22] J. Schwinger, *The Theory of Obstacles in Resonant Cavities and Wave Guides*, MIT Radiation Laboratory Report 43-34 (May 21, 1943), http://kirkmcd.princeton.edu/examples/EM/schwinger_mit_43-34.pdf

[23] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893), http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_sec_359-368.pdf

[24] K.T. McDonald, *Wire Polarizers* (Sept. 21, 2011), http://kirkmcd.princeton.edu/examples/polarizer.pdf