Polarization Dependence of Emissivity

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1 Problem

Deduce the emissive power of radiation of frequency ν into vacuum at angle θ to the normal to the surface of a good conductor at temperature T, for polarization both parallel and perpendicular to the plane of emission.

2 Solution

The solution is adapted from ref. [1] (see also [2]), and finds application in the calibration of the polarization dependence of detectors for cosmic microwave background radiation [3, 4].

Recall Kirchhoff's law of heat radiation (as clarified by Planck [5]) that,

$$\frac{P_\nu}{A_\nu} = K(\nu, T) = \frac{h\nu^3/c^2}{e^{h\nu/kT} - 1}, \tag{1}$$

where P_ν is the emissive power per unit area per unit frequency interval (emissivity) and,

$$A_\nu = 1 - \mathcal{R} = 1 - \left|\frac{E_{0r}}{E_{0i}}\right|^2, \tag{2}$$

is the absorption coefficient ($0 \leq A_\nu \leq 1$), c is the speed of light, h is Plank's constant and k is Boltzmann's constant. Also recall the Fresnel equations of reflection that,

$$\left.\frac{E_{0r}}{E_{0i}}\right|_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}, \qquad \left.\frac{E_{0r}}{E_{0i}}\right|_\parallel = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}, \tag{3}$$

where i, r, and t label the incident, reflected, and transmitted waves, respectively.

The solution is based on the fact that eq. (1) holds separately for each polarization of the emitted radiation, and is also independent of the angle of the radiation. This result is implicit in Planck's derivation [5] of Kirchhoff's law of radiation, and is stated explicitly in [6]. That law describes the thermodynamic equilibrium of radiation emitted and absorbed throughout a volume. The emissivity P_v and the absorption coefficient A_ν can depend on the polarization of the radiation and on the angle of the radiation, but the definitions of polarization parallel and perpendicular to a plane of emission, and of angle relative to the normal to a surface element, are local, while the energy conservation relation $P_\nu = A_\nu K(\nu, T)$ is global. A "ray" of radiation whose polarization can be described as parallel to the plane of emission is, in general, a mixture of parallel and perpendicular polarization from the point of view of the absorption process. Similarly, the angles of emission and absorption of a ray are different in general. Thus, the concepts of parallel and perpendicular polarization and of the angle of the radiation are not well defined after integrating over the entire volume. Thermodynamic equilibrium can exist only if a single spectral intensity function $K(\nu, T)$ holds independent of polarization and of angle.

All that remains is to evaluate the reflection coefficients $\mathcal{R}_\perp$ and $\mathcal{R}_\parallel$ for the two polarizations at a vacuum-metal interface. These are well known [1, 2, 7], but we derive them for completeness.

To use the Fresnel equations (3), we need expressions for $\sin\theta_t$ and $\cos\theta_t$. The boundary condition that the phase of the wave be continuous across the vacuum-metal interface leads, as is well known, to the general form of Snell's law,

$$k_i \sin\theta_i = k_t \sin\theta_t, \tag{4}$$

where $k = 2\pi/\lambda$ is the wave number. Then,

$$\cos\theta_t = \sqrt{1 - \frac{k_i^2}{k_t^2}\sin^2\theta_i}. \tag{5}$$

To determine the relation between wave numbers k_i and k_t in vacuum and in the conductor, we consider a plane wave of angular frequency $\omega = 2\pi\nu$ and complex wave vector $\mathbf{k}$,

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k_t}\cdot\mathbf{r} - \omega t)}, \tag{6}$$

which propagates in a conducting medium with dielectric constant ϵ, permeability μ, and conductivity σ. The wave equation for the electric field in such a medium is (in Gaussian units),

$$\nabla^2\mathbf{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{4\pi\mu\sigma}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{7}$$

where c is the speed of light. We find the dispersion relation for the wave vector k_t on inserting eq. (6) in eq. (7),

$$k_t^2 = \epsilon\mu\frac{\omega^2}{c^2} + i\frac{4\pi\sigma\mu\omega}{c^2}. \tag{8}$$

For a good conductor, the second term of eq. (8) is much larger than the first, so we write,

$$k_t \approx \frac{\sqrt{2\pi\sigma\mu\omega}}{c}(1+i) = \frac{1+i}{d} = \frac{2}{d(1-i)}, \tag{9}$$

where,

$$d = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} \ll \lambda, \tag{10}$$

is the frequency-dependent skin depth. Of course, on setting $\epsilon = 1 = \mu$ and $\sigma = 0$ we obtain expressions that hold in vacuum, where $k_i = \omega/c$.

We see that for a good conductor $|k_t| \gg k_i$, so according to eq. (5) we may take $\cos\theta_t \approx 1$ to first order of accuracy in the small ratio d/λ. Then the first of the Fresnel equations becomes,

$$\left.\frac{E_{0r}}{E_{0i}}\right|_\perp = \frac{\cos\theta_i \sin\theta_t/\sin\theta_i - 1}{\cos\theta_i \sin\theta_t/\sin\theta_i + 1} = \frac{(k_i/k_t)\cos\theta_i - 1}{(k_i/k_t)\cos\theta_i + 1} \approx \frac{(\pi d/\lambda)(1-i)\cos\theta_i - 1}{(\pi d/\lambda)(1-i)\cos\theta_i + 1}, \tag{11}$$

and the reflection coefficient is approximated by,

$$\mathcal{R}_\perp = \left|\frac{E_{0r}}{E_{0i}}\right|_\perp^2 \approx 1 - \frac{4\pi d}{\lambda}\cos\theta_i = 1 - 2\cos\theta_i\sqrt{\frac{\nu}{\sigma}}. \tag{12}$$

For the other polarization, we see that,

$$\left.\frac{E_{0r}}{E_{0i}}\right|_\parallel = \left.\frac{E_{0r}}{E_{0i}}\right|_\perp \frac{\cos(\theta_i+\theta_t)}{\cos(\theta_i-\theta_t)} \approx \left.\frac{E_{0r}}{E_{0i}}\right|_\perp \frac{\cos\theta_i - (\pi d/\lambda)(1-i)\sin^2\theta_i}{\cos\theta_i + (\pi d/\lambda)(1-i)\sin^2\theta_i}, \tag{13}$$

so that,

$$\mathcal{R}_\parallel \approx \mathcal{R}_\perp\left(1 - \frac{4\pi d}{\lambda}\frac{\sin^2\theta_i}{\cos\theta_i}\right) \approx 1 - \frac{4\pi d}{\lambda\cos\theta_i} = 1 - \frac{2}{\cos\theta_i}\sqrt{\frac{\nu}{\sigma}}. \tag{14}$$

An expression for $\mathcal{R}_\parallel$ valid to second order in d/λ has been given in [7]. For θ_i near 90°, $\mathcal{R}_\perp \approx 1$, but eq. (14) for $\mathcal{R}_\parallel$ is not accurate. Writing $\theta_i = \pi/2 - \vartheta_i$ with $\vartheta_i \ll 1$, eq. (13) becomes,

$$\left.\frac{E_{0r}}{E_{0i}}\right|_\parallel \approx \frac{\vartheta_i - (\pi d/\lambda)(1-i)}{\vartheta_i + (\pi d/\lambda)(1-i)}. \tag{15}$$

For $\theta_i = \pi/2$, $\mathcal{R}_\parallel = 1$, and $\mathcal{R}_{\parallel,\mathrm{min}} = (5-\sqrt{2})/(5+\sqrt{2}) = 0.58$ for $\vartheta_i = 2\sqrt{2}\pi d/\lambda$.

Finally, combining eqs. (1), (2), (12) and (14) we have,

$$P_{\nu\perp} \approx \frac{4\pi d\cos\theta}{\lambda^3}\frac{h\nu}{e^{h\nu/kT}-1}, \qquad P_{\nu\parallel} \approx \frac{4\pi d}{\lambda^3\cos\theta}\frac{h\nu}{e^{h\nu/kT}-1}, \tag{16}$$

and,

$$\frac{P_{\nu\perp}}{P_{\nu\parallel}} = \cos^2\theta \tag{17}$$

for the emissivities at angle θ such that $\cos\theta \gg d/\lambda$.

The conductivity σ that appears in eq. (16) can be taken as the DC conductivity so long as the wavelength exceeds 10 μm [1]. If in addition $h\nu \ll kT$, then eq. (16) can be written,

$$P_{\nu\perp} \approx \frac{4\pi d\ kT\cos\theta}{\lambda^3}, \qquad P_{\nu\parallel} \approx \frac{4\pi d\ kT}{\lambda^3\cos\theta}, \tag{18}$$

in terms of the skin depth d.

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