

Dumbbell in Orbit

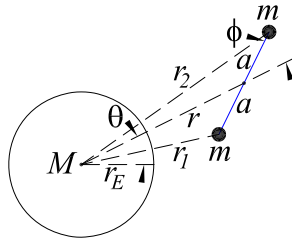
Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(April 9, 2021)

1 Problem

Discuss the stability of the orbital motion of a dumbbell about a massive force center M (such as the Earth) supposing the dumbbell consists of two masses $m \ll M$ joined by a massless, rigid rod of length $2a$, where a is not necessarily small compared to the distance r of the center of the dumbbell from the force center. It suffices to consider only motion in a plane.



2 Solution

For steady motion in a plane, and coordinates r , θ and ϕ as in the figure above, there exist equilibrium configurations with angle $\phi = 0$ and $\pi/2$, with equilibrium values r_0 and $\dot{\theta}_0$ related by Kepler's 3rd law (at least for $a \ll r_0$).¹ It seems unlikely that the equilibrium with $\phi = \pi/2$ is stable, while for $a \ll r_0$ we anticipate that the equilibrium with $\phi_0 = 0$ is stable. However, the case of a comparable to r_0 and $\phi_0 = 0$ is not immediately evident.

We would like to develop an effective potential V_{eff} for the general, planar motion, such that the sign of $\partial^2 V_{\text{eff}}(r_0)/\partial r^2$ determines the stability of small oscillations about the equilibria. However, the angular momentum of the dumbbell about the force center is not a conserved quantity, so we cannot use the form of the effective potential for orbits of spherical/point masses.

We note that the kinetic energy of the dumbbell is,

$$T = m(\dot{r}^2 + r^2\dot{\theta}^2) + ma^2(\dot{\theta} + \dot{\phi})^2, \quad (1)$$

and its potential energy is,

$$V(r, \phi) = -GMm \left(\frac{1}{r_1} + \frac{1}{r_2} \right), \quad r_{1,2}^2(r, \phi) = r^2 \mp 2ar \cos \phi + a^2. \quad (2)$$

The Lagrangian $\mathcal{L} = T - V$ does not depend explicitly on time, nor on coordinate θ , so the Hamiltonian/energy of the system is constant, as is the generalized momentum,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2m(r^2 + a^2)\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{2m(r^2 + a^2)}. \quad (3)$$

¹There also exists an equilibrium for the dumbbell perpendicular to the plane of the orbit of the center of mass, and we anticipate that this is unstable.

With this, we can write the conserved energy as,

$$E = T + V = mr^2 + \frac{mr^2 p_\theta^2}{4m^2(r^2 + a^2)^2} + ma^2 \left(\frac{p_\theta^2}{4m^2(r^2 + a^2)^2} + \dot{\phi}^2 \right) + V(r, \phi)$$

$$\equiv mr^2 + ma^2 \dot{\phi}^2 + V_{\text{eff}}(r, \phi), \quad V_{\text{eff}} = \frac{p_\theta^2}{4m(r^2 + a^2)} + V, \quad (4)$$

$$\frac{\partial V_{\text{eff}}}{\partial r} = -\frac{rp_\theta^2}{2m(r^2 + a^2)^2} + GMm \left(\frac{r - a \cos \phi}{r_1^3} + \frac{r + a \cos \phi}{r_2^3} \right), \quad (5)$$

$$\frac{\partial^2 V_{\text{eff}}}{\partial r^2} = -\frac{p_\theta^2}{2m(r^2 + a^2)^2} + \frac{2r^2 p_\theta^2}{m(r^2 + a^2)^3}$$

$$+ GMm \left(\frac{1}{r_1^3} + \frac{1}{r_2^3} - \frac{3(r - a \cos \phi)^2}{r_1^5} + \frac{3(r + a \cos \phi)^2}{r_2^5} \right), \quad (6)$$

$$\frac{\partial V_{\text{eff}}}{\partial \phi} = GMm \left(\frac{ar \sin \phi}{r_1^3} - \frac{ar \sin \phi}{r_2^3} \right), \quad (7)$$

$$\frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} = GMm \left(\frac{ar \cos \phi}{r_1^3} - \frac{ar \cos \phi}{r_2^3} - \frac{3a^2 r^2 \sin^2 \phi}{r_1^5} - \frac{3a^2 r^2 \sin^2 \phi}{r_2^5} \right). \quad (8)$$

The equations of motion for r and ϕ are,

$$\frac{\partial E}{\partial r} = 0 = 2m\ddot{r} + \frac{\partial V_{\text{eff}}}{\partial r} \quad \frac{\partial E}{\partial \phi} = 0 = 2ma^2 \ddot{\phi} + \frac{\partial V_{\text{eff}}}{\partial \phi}. \quad (9)$$

The equilibria are related by $\ddot{r} = 0 = \ddot{\phi} = 0$, and hence,

$$\frac{\partial V_{\text{eff}}(r_0, \phi_0)}{\partial r} = 0 = \frac{\partial V_{\text{eff}}(r_0, \phi_0)}{\partial \phi}. \quad (10)$$

Also, the motion for small departures from equilibrium is springlike (stable) if,

$$\frac{\partial^2 V_{\text{eff}}(r_0, \phi_0)}{\partial r^2} > 0, \quad \frac{\partial^2 V_{\text{eff}}(r_0, \phi_0)}{\partial \phi^2} > 0. \quad (11)$$

From eqs. (2) and (10) we have, noting that the equilibrium angular velocity of the center of mass of the dumbell is,

$$\dot{\theta}_0 \equiv \Omega = p_\theta / 2m(r_0^2 + a^2), \quad p_\theta = 2m\Omega(r_0^2 + a^2), \quad (12)$$

$$\frac{\partial V_{\text{eff}}(r_0, \phi_0)}{\partial r} = 0 = -2mr_0\Omega^2 + GMm \left(\frac{r_0 - a \cos \theta_0}{r_{1,0}^3} + \frac{r_0 + a \cos \theta_0}{r_{2,0}^3} \right), \quad (13)$$

$$\frac{\partial V_{\text{eff}}(r_0, \phi_0)}{\partial \phi} = 0 = GMm \left(\frac{a \sin \theta_0}{r_{1,0}^3} - \frac{a \sin \theta_0}{r_{2,0}^3} \right). \quad (14)$$

Equation (14) indicates that equilibria exist for $\theta_0 = 0$ with $r_{1,0} = r_0 - a$ and $r_{2,0} = r_0 + a$, and $\pi/2$ with $r_{1,0} = r_{2,0} = \sqrt{r_0^2 + a^2}$. For $\theta_0 = 0$, eq. (13) tells us that,

$$r_0 = \frac{GM}{2\Omega^2} \left(\frac{1}{(r_0 - a)^2} + \frac{1}{(r_0 + a)^2} \right) = \frac{GM}{\Omega^2} \frac{r_0^2 + a^2}{(r_0^2 - a^2)^2} \quad (\theta_0 = 0), \quad (15)$$

and for $\theta_0 = \pi/2$,

$$r_0 = \frac{GM}{2\Omega^2} \frac{2r_0}{(r_0^2 + a^2)^{3/2}}, \quad GM = \Omega^2(r_0^2 + a^2)^{3/2}, \quad (\theta_0 = \pi/2). \quad (16)$$

As to stability of the equilibria, we see from eq. (8) that $\partial^2 V_{\text{eff}}(r_0, \phi_0)/\partial\phi^2 > 0$ for $\phi_0 = 0$, but is negative for $\phi_0 = \pi/2$. Hence, the equilibrium at $\phi_0 = \pi/2$ is not stable.

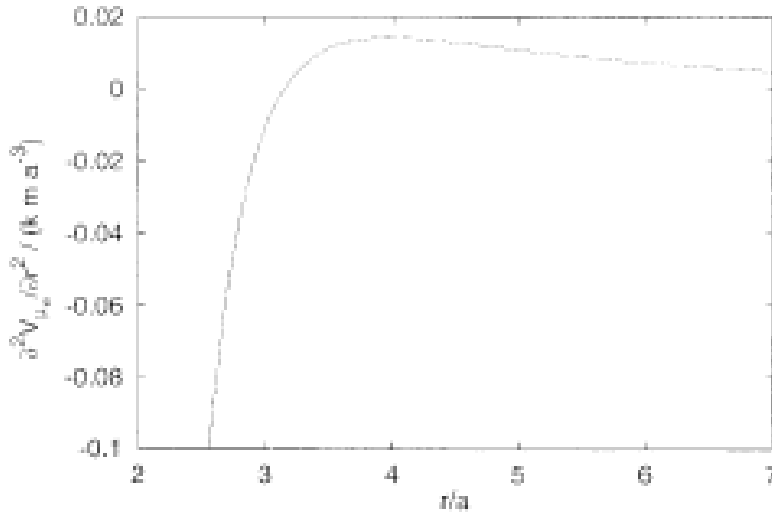
For what it's worth, eqs. (6), (12) and (16) tell us that,

$$\begin{aligned} \frac{\partial^2 V_{\text{eff}}(r_0, \pi/2)}{\partial r^2} &= -\frac{p_\theta^2}{2m(r_0^2 + a^2)^2} + \frac{2r_0^2 p_\theta^2}{m(r_0^2 + a^2)^3} + \frac{2GMm}{(r_0^2 + a^2)^{3/2}} \\ &= -\frac{2m\Omega^2(r_0^2 + a^2)^2}{(r_0^2 + a^2)^2} + \frac{8m\Omega^2 r_0^2}{r_0^2 + a^2} + 2m\Omega^2 = \frac{8m\Omega^2 r_0^2}{r_0^2 + a^2} > 0 \quad (\phi_0 = \pi/2). \end{aligned} \quad (17)$$

Finally, for $\phi_0 = 0$, using eq. (15),

$$\begin{aligned} \frac{\partial^2 V_{\text{eff}}(r_0, 0)}{\partial r^2} &= -\frac{p_\theta^2}{2m(r_0^2 + a^2)^2} + \frac{2r_0^2 p_\theta^2}{m(r_0^2 + a^2)^3} \\ &\quad + GMm \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} - \frac{3(r_0 - a)^2}{(r_0 - a)^5} + \frac{3(r_0 + a)^2}{(r_0 + a)^5} \right), \\ &= -2m\Omega^2 + \frac{8mr_0^2\Omega^2}{r_0^2 + a^2} - 2GMm \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} \right) \\ &= -2m\Omega^2 + \frac{8mr_0^2\Omega^2}{r_0^2 + a^2} - \frac{2m\Omega^2 r_0(r_0^2 - a^2)^2}{r_0^2 + a^2} \left(\frac{1}{(r_0 - a)^3} + \frac{1}{(r_0 + a)^3} \right) \quad (\phi_0 = 0). \end{aligned} \quad (18)$$

For $a \ll r$, $\partial^2 V_{\text{eff}}(r_0, 0)/\partial r^2 \approx 2m\Omega^2 > 0$, and the equilibrium at $\theta_0 = 0$ is stable. However, for a large enough, the third, negative term in the last form of eq. (18) dominates, and the equilibrium at $\phi_0 = 0$ is unstable. A numerical computation indicates that this equilibrium is unstable for $a > r_0/3$, as shown in the figure below, from [1].



The third equilibrium, with the dumbbell perpendicular to the plane of the orbit of its center of mass, is discussed in [1] and shown to be unstable.

2.1 Skyhook

A variant of a dumbbell satellite is the “skyhook”, a long, rigid rod in a geosynchronous orbit, with its lower end just above the surface of the Earth.² For this, $a \approx 12r_E$ and $r_0 \approx 13r_E$, so we infer that the skyhook is unstable.

2.2 Pendulum in Orbit

Another related problem is a pendulum in orbit, connected to a much larger mass also in orbit [6]-[8].

References

- [1] M. Krupa, A. Steindl and H. Troger, *Stability of Relative Equilibria. Part II: Dumbell Satellites*, *Meccanica* **35**, 353 (2001),
http://kirkmcd.princeton.edu/examples/mechanics/krupa_meccanica_35_353_01.pdf
- [2] J.D. Isaacs *et al.*, *Satellite Elongation into a True “Sky-Hook”*, *Science* **151**, 682 (1966),
http://kirkmcd.princeton.edu/examples/mechanics/isaacs_science_151_682_66.pdf
- [3] J. Gersten, H. Soodak and M. Tiersten, *Jack and the skyhook: The beanstalk revisited*, *Am. J. Phys.* **49**, 118 (1981),
http://kirkmcd.princeton.edu/examples/mechanics/gersten_ajp_49_118_81.pdf
- [4] K.T. McDonald. *Phy 205 Problem Set 6* (1988), Prob. 3(a),
<http://kirkmcd.princeton.edu/examples/ph205set6.pdf>
- [5] Y. Chen *et al.*, *History of the Tether Concept and Tether Missions: A Review*, *IRSN A.A.* **2013**, 502973, http://kirkmcd.princeton.edu/examples/mechanics/chen_irsnaa_2013_502973.pdf
- [6] J.L. Synge, *On the Behaviour, According to Newtonian Theory, of a Plumb Line or Pendulum Attached to an Artificial Satellite*, *Proc. Roy. Irish Acad. A* **60**, 1 (1959),
http://kirkmcd.princeton.edu/examples/mechanics/synge_priaa_60_1_59.pdf
- [7] L. Blitzer, *Equilibrium and stability of a pendulum in an orbiting spaceship*, *Am. J. Phys.* **47**, 241 (1979), http://kirkmcd.princeton.edu/examples/mechanics/blitzer_ajp_47_241_79.pdf
- [8] K.T. McDonald. *Pendulum in Orbit* (Dec. 1, 2017),
<http://kirkmcd.princeton.edu/examples/pendulum.pdf>

²The concept of the skyhook became well known following the publication of [2]. See also [3]-[5].