Kinetic Energy of Conduction Electrons in the Drude Model

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1 Problem

Discuss the kinetic energy of the conduction electrons, as well as the energy they lose to Joule heating, in the Drude Model

2 Solution

Maxwell's equations were formulated by prior to our present understanding of electrical currents as due to the motion of electrons.

We follow Drude [1] (1900) in making a simple model of the conductivity σ of a metal as due to inelastic collisions at frequency $f = 1/\tau$ of the conduction electrons with the lattice of metallic ions. If the effect of a collision is to reset electron's momentum $m\dot{\mathbf{x}}$ to zero, then for frequencies such that $\omega\tau < 1$ this discrete momentum change can be represented by a velocity-dependent friction that acts continually between collisions, and the equation of motion of an electron in an electric field $\mathbf{E} = \mathbf{E}_0\, e^{-i\omega t}$ is approximately,[1]

$$m\ddot{\mathbf{x}} = -e\mathbf{E} - \frac{m\dot{\mathbf{x}}}{\tau}\,, \tag{1}$$

whose solution is,

$$\mathbf{x} = -\frac{ie\tau\mathbf{E}}{m\omega(1-i\omega\tau)}\,, \qquad \dot{\mathbf{x}} = -\frac{e\tau\mathbf{E}}{m(1-i\omega\tau)}\,, \tag{2}$$

Then, the current density $\mathbf{J}$ is given by,

$$\mathbf{J} = -Ne\dot{\mathbf{x}} = \frac{Ne^2\tau}{m(1-i\omega\tau)}\mathbf{E} = \sigma\mathbf{E} \tag{3}$$

where N ($\approx 9\times 10^{28}/\mathrm{m}^3$ for copper) is the (volume) number density of conduction electrons.[2] The frequency-dependent metallic conductivity σ has the form,

$$\sigma = \frac{Ne^2\tau}{m(1-i\omega\tau)} = \frac{\sigma_0}{1-i\omega\tau} = \frac{\epsilon_0\omega_p^2\tau}{1-i\omega\tau}\,, \tag{4}$$

[1]We do not include Lorentz' radiation reaction force, $\mathbf{F}_{\mathrm{rad}} = (\mu_0 e^2/6\pi c)\,\dot{\mathbf{a}}$, in the equation of motion (1) because the conduction electrons do not emit any net radiation. However, if we did include the radiation reaction force $-\omega^2\tau_0 m\dot{\mathbf{x}}$, the effective damping constant $1/\tau + \omega^2\tau_0$ would differ from $1/\tau$ by only a part per million at optical frequencies (and much less than this at rf frequencies). This result tells us that radiation of energy by the conduction electrons is negligible.

[2]At very high frequencies all atomic electrons participate in the current, and N is total number density of electrons ($\approx 1.2\times 10^{30}/\mathrm{m}^3$ for copper).

with,

$$\sigma_0 = \frac{Ne^2\tau}{m}, \qquad \text{and} \qquad \omega_p = \sqrt{\frac{Ne^2}{\epsilon_0 m}}, \tag{5}$$

where σ_0 ($\approx 6 \times 10^7$ mho/m for copper) is the DC conductivity, and ω_p ($\approx 10^{16}$ s^{-1} for copper) is the plasma frequency. There are three frequency regimes of interest in Drude's model, $\omega \ll 1/\tau$, $1/\tau \lesssim \omega < \omega_p$, and $\omega > \omega_p$. For copper, the characteristic collision time is $\tau = \sigma_0 m/Ne^2 \approx 2 \times 10^{-14}$ s. Thus, for radio frequencies ($\omega \approx 10^9$ s^{-1}, say, for which the wavelength is $\lambda = 2\pi c/\omega \approx 2$ m), $\omega\tau \ll 1$ and the Drude-model conductivity is well approximated by its real, DC value. For optical frequencies ($\omega \approx 4 \times 10^{15}$ s^{-1}), $\omega\tau > 1$ and Drude's model predicts that the conductivity is essentially pure imaginary, $\sigma_{\rm optical} \approx -i\epsilon_0\omega_p^2/\omega$. Drude's classical electron model of electrical conductivity is less accurate at optical than rf frequencies, and we must turn to a quantum model for better understanding of metallic conductivity in the optical regime. See, for example, secs. 86-87 of [2]. Drude's model is again rather accurate when $\omega \gg \omega_p$, but as we shall see in sec. 3.2, conductors are essentially transparent in this limit.

One significance of the small imaginary part of the conductivity (4) is that it accounts for the power associated with changes in the time-varying kinetic energy of the conduction electrons.[3] The imaginary part of the conductivity leads to a term in the current density $\mathbf{J} = \sigma\mathbf{E}$ that is out of phase with the electric field, and hence part of the power $\mathbf{J} \cdot \mathbf{E}$ that is delivered to the current $\mathbf{J}$ causes no time-averaged change in the energy of the system, as expected for the oscillatory kinetic energy of the conduction electrons.

In greater detail, if we write the electric field at some point inside the conductor as $\mathbf{E}_c\, e^{-i\omega t}$, then the physical electric field is,

$$\mathbf{E} = Re(\mathbf{E}_c\, e^{-i\omega t}) = Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t, \tag{6}$$

and the physical current density is,

$$\begin{aligned} \mathbf{J} &= Re(\sigma\mathbf{E}) = Re\left[\sigma_0 \frac{1 + i\omega\tau}{1 + \omega^2\tau^2}\mathbf{E}_c\, e^{-i\omega t}\right] \qquad (7) \\ &= \frac{Ne^2\tau}{m(1 + \omega^2\tau^2)}\{[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t] - \omega\tau[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t]\}. \end{aligned}$$

Then, the physical density of power delivered to the current density is,

$$\begin{aligned} \mathbf{J} \cdot \mathbf{E} &= \frac{\sigma_0}{1 + \omega^2\tau^2}\{[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t]^2 \qquad (8) \\ &\quad - \frac{Ne^2\omega\tau^2}{m(1 + \omega^2\tau^2)}[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t] \cdot [Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t]. \end{aligned}$$

The first term of eq. (8) is, of course, the power dissipated by Joule heating. We relate the second term to the time rate of change of the kinetic energy of the conduction electrons,

$$\frac{d}{dt}u_{\rm KE} = \frac{d}{dt}\left(\frac{Nmv^2}{2}\right) = Nm\mathbf{v} \cdot \mathbf{a}, \tag{9}$$

[3]The velocity of a conduction electron has the form $\mathbf{v}_{\rm random} + \mathbf{v}_{\rm drift}$ where $v_{\rm random} \gg v_{\rm drift}$. The total kinetic energy of these electron is $\sum m(\mathbf{v}_{\rm random} + \mathbf{v}_{\rm drift})^2/2 = \sum mv_{\rm random}^2/2 + \sum mv_{\rm drift}^2/2$. The oscillatory part of the kinetic energy is $\sum mv_{\rm drift}^2/2$, the density of which we call $u_{\rm KE}$.

by noting that the velocity of the conduction electrons is $\mathbf{v} = -\mathbf{J}/Ne$, so that their acceleration $\mathbf{a} = d\mathbf{v}/dt$ is,

$$\mathbf{a} = \frac{e\omega\tau}{m(1+\omega^2\tau^2)}\{[Re(\mathbf{E}_c)\sin\omega t - Im(\mathbf{E}_c)\cos\omega t] + \omega\tau[Re(\mathbf{E}_c)\cos\omega t + Im(\mathbf{E}_c)\sin\omega t]\}. \quad (10)$$

Thus, for $\omega\tau \ll 1$, eqs. (7) and (9)-(10) show that the second term of eq. (8) is the time rate of change of the (drift) kinetic energy of the conduction electrons (plus terms of order $\omega^2\tau^2$).[4]

Although Drude's model gives only an approximate understanding of conductors at optical frequencies, it does predict that in this regime the power dissipated by Joule heating is small compared to the power that changes the kinetic energy of the conduction electrons, and so provides some insight as to a microscopic view of very good conductors in which quasi-free electrons are the charge carriers.

References

[1] P. Drude, *Zur Elektronentheorie der Metalle*, Ann. Phys. **1**, 566 (1900), http://kirkmcd.princeton.edu/examples/EM/drude_ap_1_566_00.pdf

[2] E.M. Lifshitz and L.P. Pitaevskii, *Physical Kinetics* (Butterworth-Heinemann, 1981).

[3] K.T. McDonald, *How Much of Magnetic Energy Is Kinetic Energy?* (Sept. 12, 2009), http://kirkmcd.princeton.edu/examples/kinetic.pdf

[4]An implication is that the (drift) kinetic energy of conduction electrons is part of the "electromagnetic field" energy. In AC circuits with negligible capacitance, this field energy is largely "magnetic".

While Maxwell did call the "magnetic" field energy a "kinetic" energy, he did not consider that electric currents involve moving electric charges. See, for example, [3].