Does $\nabla \cdot \mathbf{J} = 0$ Imply $\nabla \cdot \mathbf{A} = 0$?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

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1 Problem

In electromagnetism, the condition $\nabla \cdot \mathbf{J} = 0$ on the electric-current density $\mathbf{J}$ implies that the electric charge density ρ is time independent, according to the continuity equation $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$ (conservation of electric charge), which in turn implies that $\mathbf{J}$ is time-independent (steady currents). The condition $\nabla \cdot \mathbf{J} = 0$ also implies the lines of $\mathbf{J}$ form closed loops.[1]

That is, $\nabla \cdot \mathbf{J} = 0$, for nonzero $\mathbf{J}$, implies both electrostatics and magnetostatics.[2] It is sometimes assumed that for static electromagnetism, $\nabla \cdot \mathbf{A} = 0$ (perhaps following Maxwell, Art. 617 of [5]), where $\mathbf{A}$ is the electromagnetic vector potential, which is related to the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ by,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{1}$$

in SI units, where V is the electric scalar potential.

But, does $\nabla \cdot \mathbf{J} = 0$ actually imply that $\nabla \cdot \mathbf{A} = 0$?

2 Solution

In general, the answer is NO.

One way to see this is to consider the vector potential $\mathbf{A}$ in the so-called Poincaré gauge (see sec. 9A of [6] and [7, 8, 9]),[3] where the gauge condition is $\mathbf{A}\cdot\mathbf{x} = 0$, and the potentials are computed via integrals along the line from the (arbitrary) origin to the point $\mathbf{x}$ of observation,

$$V(\mathbf{x}, t) = -\mathbf{x} \cdot \int_0^1 du\, \mathbf{E}(u\mathbf{x}, t), \qquad \mathbf{A}(\mathbf{x}, t) = -\mathbf{x} \times \int_0^1 u\, du\, \mathbf{B}(u\mathbf{x}, t). \tag{2}$$

The divergence of the Poincaré-gauge vector potential is,

$$\nabla \cdot \mathbf{A} = \mathbf{x} \cdot \int_0^1 u\, du\, \nabla \times \mathbf{B}(u\mathbf{x}, t) = \mathbf{x} \cdot \int_0^1 u\, du \left(\mu_0 \mathbf{J}(u\mathbf{x}, t) + \frac{1}{c^2}\frac{\partial \mathbf{E}(u\mathbf{x}, t)}{\partial t}\right). \tag{3}$$

[1] As is the case for lines of the magnetic field $\mathbf{B}$, which obey $\nabla \cdot \mathbf{B} = 0$, field lines of $\mathbf{J}$ (when its divergence is zero) do not necessarily form simple (one-turn) loops. But, this does not mean that the field lines can be "open-ended", as implied, for example, in [1]-[3]; a nonphysical, uniform field is the exception.

A subtler issue was discussed in [4], as to whether if the field lines make an infinite number of turns they should be called "closed". The view of the present author is that they should.

[2] (Unphysical) source-free electromagnetic waves have $\mathbf{J} = 0$, and hence $\nabla \cdot \mathbf{J} = 0$ also.

[3] The Poincaré gauge is also called the multipolar gauge [10, 11].

In static examples with only azimuthal currents we have that $\nabla \cdot \mathbf{A} = 0$ in the Poincaré gauge when the origin is on the symmetry axis,[4] but for more general current densities (and for more general choice of the origin), $\nabla \cdot \mathbf{A} \neq 0$ (in this gauge).

2.1 YES, If the Vector Potential Can Be Set to Zero at Infinity

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As noted, for example, in sec. 5.4.1 of [13][5] and on p. 53 of [14], if we can enforce the auxiliary condition that the vector potential vanishes at infinity (in all directions) for steady currents, then it follows that $\nabla \cdot \mathbf{A} = 0$ everywhere.

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[4]For an example of this type, see [12].

[5]Equation (5.63) implies that $\nabla \cdot \mathbf{A} = 0$.

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