A “Preliminary” Rolling Disk Problem

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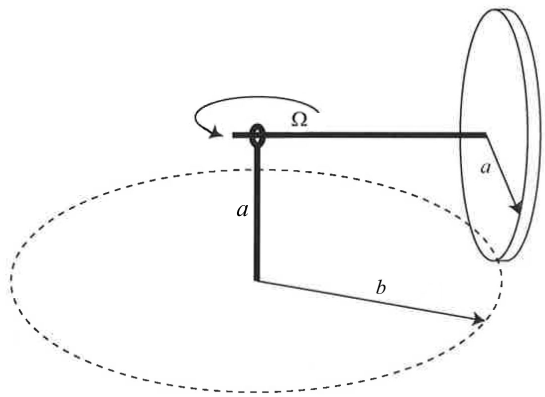
1 Problem

A thin, uniform disk of mass m and radius a rolls (without slipping?) in a horizontal circle of radius b. The (massless) axle of the disk is connected to a fixed pivot point at height a above the ground at the center of the circle. The pivot cannot exert a tension along the axle, as indicated by the loop in the pivot as sketched below.



The motion is steady, with constant angular velocity Ω about the vertical, assuming no dynamic friction due to air resistance or rolling friction at the ground or at the bearings on the axle.

What are the scalar components of the static-friction force $\mathbf{F}_S$ of the ground on the disk and of the force $\mathbf{F}_P$ of the pivot on the axle?

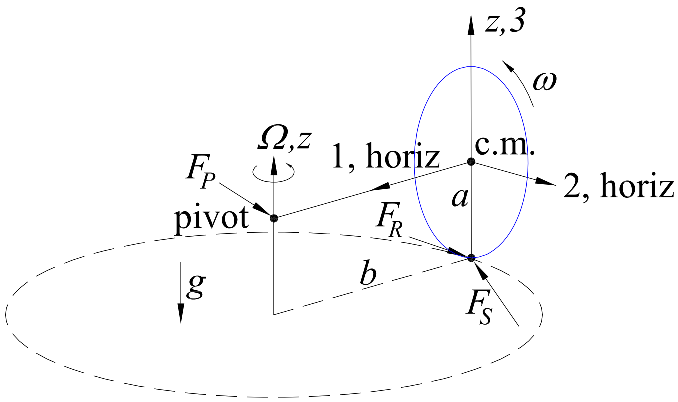
What can be said if rolling friction $\mathbf{F}_R$ at the point of contact of the disk with the ground is modeled as a force of magnitude $F_R = \mu N$, opposing the direction of motion, where N is the normal force of the ground on the disk (the vertical component of $\mathbf{F}_S$), and μ is a constant?

This problem is adapted from Prob. A.2 of the Princeton Graduate Preliminary Examination of Jan. 2020 [1]. Possibly it was based on Prob. 9.51 of [2].

2 Solution

2.1 Equations of Motion

In addition to the vertical z-axis, we consider a set of principle axes, $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ of the disk, with horizontal axis $\hat{\mathbf{1}}$ pointing from the center of the disk to the pivot point along the axle of the disk, horizontal axis $\hat{\mathbf{2}}$ in the vertical plane of the disk, and vertical axis $\hat{\mathbf{3}} = \hat{\mathbf{z}}$ also in the plane of the disk, as sketched on the next page.



Rolling friction $\mathbf{F}_R$ has only a 2-component, while static (or sliding) friction $\mathbf{F}_S$ and the pivot force $\mathbf{F}_P$ can point in any direction. *For now, we suppose that $F_{P,1}$ could be nonzero.*

The equation of motion of the center of mass of the disk, which is at the center of the disk in the approximation of a massless axle, is simply,

$$m\frac{d\mathbf{v}_{\rm cm}}{dt} = \mathbf{F}_P + \mathbf{F}_R + \mathbf{F}_S - mg\,\hat{\mathbf{z}}, \tag{1}$$

if we ignore the air resistance and rolling friction at the bearing of the disk.

Since the center of a disk moves in a horizontal circle of radius b with angular velocity Ω and center of mass velocity $\mathbf{v}_{\rm cm} = -\Omega\, b\,\hat{\mathbf{2}}$. The acceleration $d\mathbf{v}_{\rm cm}/d/t$ of the c.m. has $\hat{\mathbf{1}}$-component $= m\Omega^2\, b =$ centripetal acceleration, $\hat{\mathbf{2}}$-component $= -\dot{\Omega}\, b$, while the $\hat{\mathbf{3}}$-component is zero.

The disk has angular velocity ω about $\hat{\mathbf{1}}$, the axle of the system. If we suppose the disk rolls without slipping and $\Omega > 0$, then $\omega > 0$ and,

$$v_{\rm cm} = \omega a, \qquad \mathbf{v}_{\rm cm} = -\omega a\,\hat{\mathbf{2}}, \tag{2}$$

and hence,

$$\omega = \frac{b}{a}\Omega. \tag{3}$$

However, we defer use of relation (3) so that the system could be considered a spinning top/gyroscopic with one point fixed if $\mathbf{F}_R$ and $\mathbf{F}_S$ were zero.

The scalar components of eq. (1) can now be written as,

$$m\Omega^2\, b = F_{P,1} + F_{S,1}, \qquad -m\dot{\Omega}\, b = F_{P,2} + F_R + F_{S,2}, \qquad 0 = F_{P,3} + F_{S,3} - mg. \tag{4}$$

The center of the disk has angular velocity Ω about the pivot point, so the total angular velocity is,

$$\boldsymbol{\omega}_{\rm tot} = \omega\,\hat{\mathbf{1}} + \Omega\,\hat{\mathbf{3}}. \tag{5}$$

The principal axes $\mathbf{i}$ have angular velocity $\Omega\,\hat{\mathbf{3}}$, such that $d\hat{\mathbf{i}}/dt = \Omega\,\hat{\mathbf{3}} \times \hat{\mathbf{i}}$,

$$\frac{d\hat{\mathbf{1}}}{dt} = \Omega\,\hat{\mathbf{2}}, \qquad \frac{d\hat{\mathbf{2}}}{dt} = -\Omega\,\hat{\mathbf{1}}, \qquad \frac{d\hat{\mathbf{3}}}{dt} = 0. \tag{6}$$

The disk has moments of inertia, with respect to its center of mass $I_1 = kma^2 = 2I_2 = 2I_3$ due to the axial symmetry of the disk (with $k = 1/2$ for a uniform disk). The angular momentum of the disk with respect to its center of mass is,

$$\mathbf{L}_{\rm cm} = I_1\,\omega\,\hat{\mathbf{1}} + I_2\,\Omega\,\hat{\mathbf{3}} = I_1\,\omega\,\hat{\mathbf{1}} + \frac{I_1\,\Omega}{2}\,\hat{\mathbf{3}}. \tag{7}$$

Then, the angular equation of motion with respect to the c.m. of the disk is,[1]

$$\begin{aligned} \frac{d\mathbf{L}_{\rm cm}}{dt} &= I_1\,\dot\omega\,\hat{\mathbf{1}} + I_1\,\omega\,\Omega\,\hat{\mathbf{2}} + \frac{I_1\,\dot\Omega}{2}\,\hat{\mathbf{3}} = \boldsymbol{\tau}_{\rm cm} \\ &= b\,\hat{\mathbf{1}}\times\mathbf{F}_P - a\,\hat{\mathbf{3}}\times\mathbf{F}_R - a\,\hat{\mathbf{3}}\times\mathbf{F}_S \\ &= bF_{P,2}\,\hat{\mathbf{3}} - bF_{P,3}\,\hat{\mathbf{2}} + aF_R\,\hat{\mathbf{1}} - aF_{S,1}\,\hat{\mathbf{2}} + aF_{S,2}\,\hat{\mathbf{1}}. \end{aligned} \tag{8}$$

The three scalar components of this equation of motion are,

$$I_1\,\dot\omega = aF_R + aF_{S,2}, \qquad I_1\,\omega\,\Omega = -aF_{S,1} - bF_{P,3}, \qquad I_1\,\dot\Omega = 2bF_{P,2}. \tag{9}$$

2.2 Steady Motion with No Rolling Friction

For steady motion, $\dot\omega = 0 = \dot\Omega$, and there can be no rolling friction F_R (which dissipates energy). Then, eqs. (4) and (9) become,

$$m\Omega^2\,b = F_{P,1} + F_{,}, \qquad 0 = F_{P,2} + F_{S,2}, \qquad F_{P,3} + F_{S,3} = mg, \tag{10}$$

$$0 = F_{S,2}, \qquad I_1\,\omega\,\Omega = -aF_{S,1} - bF_{P,3}, \qquad 0 = F_{P,2}. \tag{11}$$

2.2.1 Gyroscopic Motion

One possibility is that there is that $\mathbf{F}_R = 0 = \mathbf{F}_S$, and the motion is gyroscopic precession,

$$F_{P,1} = m\Omega^2 b, \qquad F_{P2} = 0, \qquad F_{P,3} = mg, \qquad \Omega = -\frac{mgb}{I_1\omega}\,. \tag{12}$$

For gyroscopic motion with $\omega > 0$, Ω is negative, and opposite to that shown in the figure on p. 1. The normal force N is zero in this case, and the horizontal surface need not be present.

2.2.2 $F_{P,1} = 0$ and Rolling without Slipping

If we now suppose that $F_{P,1} = 0$, *i.e.*, the pivot exerts no force along the axle, the first of eq. (10) tells us that,

$$F_{S,1} = m\Omega^2\,b, \tag{13}$$

[1] The claim in the statement of this problem in [1] that the pivot/"joint can apply forces but no torques" is misleading. Probably it was meant that the "joint" applies no axial torque on the axle (which would change the angular velocity ω of the disk). The force $\mathbf{F}_P$ at the pivot/joint does apply nonzero torque about the c.m.

and the second of eq. (11), together with the rolling constraint (3), $\omega = b\,\Omega/a$, implies that

$$F_{P,3} = -\frac{a}{b}F_{S,1} - \frac{I_1\omega\,\Omega}{b} = -m\,\Omega^2 a - \frac{I_1\Omega^2}{a} = -\frac{I_1 + ma^2}{a}\Omega^2 = -(k+1)ma\Omega^2. \qquad (14)$$

Then, the third of eq. (10) indicates that the (upward) normal force $N = F_{S,3}$ on the disk is,

$$N = F_{S,3} = mg - F_{P,3} = mg + (k+1)ma\Omega^2. \qquad (15)$$

If we had not assumed that $F_{P,1} = 0$, but only that the disk rolls without slipping, the normal force N would not be well determined by the above analysis.

2.3 Effect of Rolling Friction

If we again suppose that $F_{P,1} = 0$, eqs. (4) and (9) including rolling friction F_R become,

$$m\Omega^2 b = F_{S,1}, \qquad -m\dot\Omega\,b = F_{P,2} + F_R, \qquad F_{P,3} + F_{S,3} = mg, \quad (16)$$

$$I_1\dot\omega = \frac{I_1 b\,\dot\Omega}{a} = aF_R + aF_{S,2}, \qquad I_1\,\omega\,\Omega = \frac{I_1 b\,\Omega^2}{a} = -aF_{S,1} - bF_{P,3}, \qquad I_1\dot\Omega = 2bF_{P,2}, \quad (17)$$

assuming also that the rolling constraint (3) holds.

The presence of rolling friction does not affect the argument in sec. 2.2.2 that led to eq. (15), so the normal force N on the disk is again given by that equation. Then, in the model that $\mathbf{F}_R = \mu N\,\hat{\mathbf{2}}$, the second of eq. (16) and the third of eq. (17) tell us that

$$-m\dot\Omega\,b = \frac{I_1\dot\Omega}{2b} + F_R, \qquad \dot\Omega = -\frac{2b}{I_1 + 2mb^2}\mu N = -\frac{2\mu b}{ka^2 + 2b^2}\left(g + (k+1)a\Omega^2\right). \qquad (18)$$

which can be integrated numerically to find $\Omega(t)$. For small initial Ω_0, the angular velocity Ω drops to zero in time $t \approx (ka^2 + 2b^2)\Omega_0/2\mu bg$.

Now, $F_R(t) = \mu N(t)$ can be obtained from eq. (15), $F_{S,1}(t)$ can be obtained from the first of eq. (16), $F_{S,2}(t)$ can be obtained from the first of eq. (17), $F_{P,2}(t)$ can be obtained from the third of eq. (17), $F_{P,3}(t)$ can be obtained from the second of eq. (17), and finally $F_{S,3}(t)$ can be obtained from the third of eq. (16).

References

[1] *Princeton Physics Graduate Preliminary Examination, Part I* (Jan. 9, 2020),
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[2] D. Morin, *Introduction to Classical Mechanics* (Cambridge U. Press, 2008),
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[3] K.T. McDonald, *Comments on Torque Analyses* (Apr. 28, 2019),
http://kirkmcd.princeton.edu/examples/torque.pdf

[4] K.T. McDonald, *Ph 205, Lecture 19*,
http://kirkmcd.princeton.edu/examples/Ph205/ph205l19.pdf