Magnetic Forces Can Do Work

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1 Problem

Discuss the motion, and the work-energy relation, for a nonconducting, uniform disk of mass M, radius R with n charges q of mass m fixed to its rim at equal intervals.[1] The disk has initial angular velocity ω_0 about its axis, the z-axis, where $v_\phi = \omega a \ll c$, and c is the speed of light, and the center of the disk is initially at the origin. The disk is immersed in a magnetic field with axial symmetry (maintained constant in time by distant sources, perhaps permanent magnets), with field along the z-axis given by,

$$B_z(z) = B_0 + B_1 z. \tag{1}$$

2 Solution

This problem provides an illustration of what can be meant by the statement that "magnetic forces do no work", as discussed in sec. 5.1.2 of [1].[2,3,4]

We show this statement holds only if "magnetic forces" means the effect of both magnetic Lorentz forces and torques on electric charges (and not on their intrinsic magnetic moments), and not the magnetic force $(\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\mathbf{B}$ that equals the Lorentz force on the magnetic moment $\boldsymbol{\mu}$ of the system.

See [7, 8] for examples where the magnetic force $-\boldsymbol{\nabla}(-\boldsymbol{\mu} \cdot \mathbf{B}) = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B})$ does work on a permanent magnetic moment $\boldsymbol{\mu}$. Of course, the fact that if a refrigerator magnet is held close to the (iron) surface of the refrigerator, it will be pulled out of your hand, confirming that magnetic forces can do work.

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[1] All fundamental electric charges in Nature carry small magnetic moments. However, you may neglect the magnetic moment of the charges in this problem. Also, you may assume that mass M include that mass nm of the charges, whose mass density is the same as that of the rest of the disk.

[2] *Oct. 22, 2021.* Griffiths' argument, as applied to macroscopic analyses where considerations of $\mathbf{J} \times \mathbf{B}$ are relevant, involves something of a play on words, in that these forces do work in the Newtonian sense whenever the conductor has a velocity component perpendicular to $\mathbf{J}$. However, the source of the energy is not the magnetic field itself, but rather is a "battery" (including an electric generator or a capacitor), or "mechanical". Griffiths proposes that a force does not do work unless that force is somehow the source or the energy transferred, which view is inconsistent with usage in mechanics. He illustrates his wordplay in Ex. 5.3, p. 209 of [1].

[3] Other examples involving a "battery" where magnetic forces do work include Ex. 7.6, p. 304 of [1], and [2]-[5].

[4] *Nov. 21, 2022.* Wald, p. 72 of [6], argues that the phrase "magnetic fields do no work" means that the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ transfer energy to matter with electric current density $\mathbf{J}$ at rate $\mathbf{J} \cdot \mathbf{E}$, which does not involve the $\mathbf{B}$ field.

2.1 The External Magnetic Field

A static magnetic field with axial symmetry in some region $r < r_0$ and $|z| < z_0$ in cylindrical coordinates (r, ϕ, z) can be determined from knowledge of $B_z(0, 0, z)$ [9], which is an unusual "boundary condition". The determination follows from the (static) Maxwell equations for magnetic fields with axial symmetry in regions of zero current,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z}\,, \qquad \left(\text{and} \qquad \boldsymbol{\nabla} \times \mathbf{B} = 0, \qquad i.e., \qquad \frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r}\right). \tag{2}$$

In the present case $\partial B_z(0, 0, z)/\partial z = B_1$. If we assume this to be independent of r (which happens to be true), then eq. (2) integrates to,

$$B_r(r, \phi, z) = -\frac{B_1 r}{2}\,. \tag{3}$$

If $B_z(0, 0, z)$ had higher nonzero derivatives than the first, the forms for B_r and B_z would be more complicated [9], but here the external field is given by eq. (3) and,

$$B_z(r, \phi, z) = B_0 + B_1 z. \tag{4}$$

These forms obey $\boldsymbol{\nabla} \times \mathbf{B} = 0$.

2.2 Analysis via the Lorentz Force

The motion of a rigid body is determined by the external forces and torques upon it. In the present example, the only external force is the Lorentz force on the n charges. The force on one of these charges is,

$$\begin{aligned} \mathbf{F}_{q,\text{ext}} &= q\frac{\mathbf{v_q}}{c} \times \mathbf{B}_{\text{ext}} = \frac{qv_\phi B_z}{c}\,\hat{\mathbf{r}} + \frac{qv_z B_r}{c}\,\hat{\boldsymbol{\phi}} - \frac{qv_\phi B_r}{c}\,\hat{\mathbf{z}} \\ &= \frac{q\omega R(B_0 + B_1 z)}{c}\,\hat{\mathbf{r}} - \frac{qv_z B_1 R}{2c}\,\hat{\boldsymbol{\phi}} + \frac{q\omega R^2 B_1}{2c}\,\hat{\mathbf{z}}\,. \end{aligned} \tag{5}$$

The radial component of this force does no work (as the charges are at fixed radius R). The azimuthal force, and its associated torque, try to spin the disk down, and the axial force pushes the disk in the positive z-direction. The equations of motion for the disk are,

$$\frac{MR^2\dot{\omega}}{2} = nRF_{q,\phi} = -\frac{nq\dot{z}B_1R^2}{2c}\,, \qquad \dot{\omega} = -\frac{nqB_1}{Mc}\,\dot{z}, \tag{6}$$

$$M\ddot{z} = nF_{q,z} = \frac{nq\omega R^2 B_1}{2c} \qquad \ddot{z} = \frac{nqR^2B_1}{2Mc}\,\omega. \tag{7}$$

The initial conditions are $\omega(0) = \omega_0$, $\dot{\omega}(0) = 0$, $z(0) = 0 = \dot{z}(0)$, and $\ddot{z}(0) = nq\omega_0 R^2 B_1/2Mc$.

Combining equations (6) and (7), we have,

$$\dddot{z} = -\frac{n^2q^2R^2B_1^2}{2M^2c^2}\,\dot{z} \equiv -\Omega^2\,\dot{z}, \qquad \text{with} \qquad \Omega = \frac{nqRB_1}{\sqrt{2}Mc}\,, \tag{8}$$

which integrates to

$$\ddot{z} = -\Omega^2\, z + \ddot{z}(0) = -\Omega^2 \left(z - \frac{\ddot{z}(0)}{\Omega^2} \right) = -\Omega^2 \left(z - \frac{M\omega_0 c}{nqB_1} \right) \tag{9}$$

The solution for this with $z(0) = 0 = \dot{z}(0)$ is

$$z(t) = \frac{M\omega_0 c}{nqB_1}(1 - \cos\Omega t). \tag{10}$$

Then,

$$\dot{z}(t) = \frac{M\omega_0 c}{nqB_1}\Omega \sin\Omega t = \frac{\omega_0 R}{\sqrt{2}} \sin\Omega t. \tag{11}$$

The equation of motion (6) for ω can now be written as,

$$\dot{\omega} = -\frac{nqB_1}{Mc}\, \dot{z} = -\omega_0 \Omega \sin\Omega t, \tag{12}$$

which integrates to

$$\omega(t) = \omega_0 \cos\Omega t. \tag{13}$$

The motion is oscillatory. At first, the center of mass of the disc speeds up along z while the angular velocity of the disk slows down. When the angular velocity reverses direction, the c.m. begins to slow down, and eventually reverses direction, *etc.*

The kinetic energy of the system is,

$$\mathrm{KE} = \frac{M\dot{z}^2}{2} + \frac{MR^2\omega^2}{4} = \frac{MR^2\omega_0^2 \sin^2\Omega t}{4} + \frac{MR^2\omega_0^2 \cos^2\Omega t}{4} = \frac{MR^2\omega_0^2}{4} = \mathrm{KE}_0. \tag{14}$$

The kinetic energy is constant, and no net work is done on the system. The work done by the Lorentz magnetic force on the system in changing the center-of-mass velocity is equal and opposite to the work by the Lorentz magnetic torque in changing the angular velocity.

The sum of the work done by the magnetic forces on the charges of a system (which includes work done by magnetic torques) is always zero, but the work done by the sum of the magnetic forces on the charges (which changes the translational kinetic energy) can be nonzero.

A variant on this problem would be to suppose the angular velocity ω_0 is kept constant by some mechanism that can apply torques to the disk, while permitting it to slide along the z-axis. In this case the disk would have constant linear acceleration along the z-axis, and the mechanism would have to apply an ever stronger torque to counteract the rising magnetic toque. The positive work done by the total Lorentz magnetic force in increasing the translational kinetic energy of the disk would be equal and opposite to the negative work done by the Lorentz magnetic torque.

Another variant is to suppose that the center of the disk is fixed, but that it is free to spin. In this case the azimuth force on a charge vanishes in eq. (5), and the radial and axial forces do no work because there is no radial or axial motion, and the angular velocity is constant.

2.3 Analysis via the Magnetic Moment of the Spinning Disk

Instead of emphasizing the Lorentz force law, we can give prominence to the magnetic moment of the spinning, charged disk,

$$\boldsymbol{\mu} = \frac{\sum q\mathbf{r}_q \times \mathbf{v}_q}{2c} = \frac{nq\omega R^2}{2c}\,\hat{\mathbf{z}}. \tag{15}$$

The total electromagnetic force on the disk is the sum for the Lorentz forces (5),

$$\mathbf{F} = \sum \mathbf{F}_{q,\text{ext}} \quad = \quad \frac{nq\omega R^2 B_1}{2c}\,\hat{\mathbf{z}} \;= \boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\mathbf{B}), \tag{16}$$

recalling eq. (1) for the external magnetic field.

Comparing with sec. 2.2, we see that with $\mathbf{v} = \dot{z}\,\hat{\mathbf{z}}$ as the center-of-mass velocity of the disk, then,

$$\mathbf{F}\cdot\mathbf{v} = \frac{d\text{KE}_{\text{translation}}}{dt} \neq 0. \tag{17}$$

Hence, magnetic forces can do work, if the term "magnetic forces" simply means the total Lorentz force on the system, which is the force (16) on the magnetic moment in the present case.

The torque on the n charges is,

$$\boldsymbol{\tau} = \sum \mathbf{r}_q \times \mathbf{F}_q = -\frac{nq\dot{z}B_1R^2}{2c}\,\hat{\mathbf{z}} = -\frac{\dot{z}B_1}{\omega}\,\mathbf{m} = -\frac{B_1R\tan\Omega t}{\sqrt{2}}\,\mathbf{m}. \tag{18}$$

This is not of the form $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$ that holds for magnetic dipoles which are physically small compared to the length scale for changes in $\mathbf{B}$.

References

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