1 Problem

What is the (vector) electric field strength at the center of a spherical cavity of radius $a$ in a grounded conductor if a point dipole $p$ is placed at distance $b$ $(0 < b < a)$ from the center of the cavity so as to make angle $\alpha$ to the radius vector to the dipole?

2 Solution

This problem is question 05/01 at Yakov Kantor’s Physics Quiz site, http://www.tau.ac.il/~kantor/QUIZ/

We solve via the image method for spherical geometry. Recall that a charge $q$ at radius $b < a$ results in an image charge $q'$ at $a^2/b$, such that the potential at $r = a$ vanishes,

$$\phi(r = a) = 0 = \frac{q}{a - b} + \frac{q'}{a^2/b - a} = \frac{q}{a - b} + \frac{q'b/a}{a - b},$$

and hence $q' = -qa/b$.

For the present case of a point dipole $p$, we can think of this as due to charges $\pm q$ separated by distance $\delta = q/p$, with the charge $-q$ at radius $b$ and the charge $+q$ at radius $b + \delta \cos \alpha$ as shown below.

The image transformation $r \rightarrow a^2/r$ is conformal (angle preserving), so the triangle formed by the center of the sphere and charges $\pm q$ is similar to that formed by the center and the image charges $q'$ and $q''$. Image charge $q' = +qa/b$ is at distance $a^2/b$ from the center. Hence, the distance between the two image charges is,

$$\delta' = \delta \frac{a^2}{b^2}.$$
The image charge $q''$ has value,

$$q'' = -q \frac{a}{b + \delta \cos \alpha} \approx -q \frac{a}{b} + q \delta \cos \alpha \frac{a}{b^2}. \quad (3)$$

Thus, taking the limit $\delta \to 0$, $q \to \infty$ such that $q \delta = p$, the two image charges can be thought of as a dipole of strength,

$$p' = -q \frac{a^2}{b^2} = -p \frac{a^3}{b^3}, \quad (4)$$

at angle $\alpha$ to the radius vector as shown above, plus an additional charge of magnitude,

$$q''' = p \frac{a}{b^2} \cos \alpha, \quad (5)$$

located at distance $a^2/b$ from the center. If $\alpha = \pm \pi/2$, the image of the dipole $p$ is a pure dipole.

To calculate the electric field at the center, we recall that the field of a point dipole $p$ is,

$$E = \frac{3(p \cdot \hat{r})\hat{r} - p}{r^3}, \quad (6)$$

where $\vec{r}$ points from the dipole to the observer. In the present problem, we define $\hat{z}$ to point from the center to the dipoles, so $\hat{r} = -\hat{z}$. Dipole $p$ is taken to lie in the $x$-$z$ plane, so that,

$$p = p\hat{z} \cos \alpha + p\hat{x} \sin \alpha, \quad (7)$$

and,

$$p' = p \frac{a^3}{b^3} \hat{z} \cos \alpha - p \frac{a^3}{b^3} \hat{x} \sin \alpha. \quad (8)$$

Then, the field at the center of the cavity due to the dipole, and its image dipole + extra charge $q'''$ is,

$$E(0) = \frac{3(p \cdot \hat{z})\hat{z} - p}{b^3} + \frac{3(p' \cdot \hat{z})\hat{z} - p'}{(a^2/b)^3} - \frac{q'''\hat{z}}{(a^2/b)^2}$$

$$= \frac{2p\hat{z} \cos \alpha - p\hat{x} \sin \alpha}{b^3} + \frac{2p\hat{z} \cos \alpha + p\hat{x} \sin \alpha}{a^3} - \frac{p\hat{z} \cos \alpha}{a^3}$$

$$= p\hat{z} \cos \alpha \left(\frac{2}{b^3} + \frac{1}{a^3}\right) - p\hat{x} \sin \alpha \left(\frac{1}{b^3} - \frac{1}{a^3}\right). \quad (9)$$