Early Use of Dimensional Analysis in Quantum Theory

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A recent paper [1] presented an alternative history of early quantum theory in which use of the so-called Pi theorem of dimensional analysis was made. Here, we give two examples of the early use of dimensional analysis in quantum theory.

1. We recall that Wien gave a distribution law for the spectrum of blackbody radiation in 1896 [2],

$$I(\lambda, T) = \frac{C_1}{\lambda^5} e^{-C_2/\lambda T}, \tag{1}$$

where λ is the wavelength, T is the absolute temperature, and the C_i are constants. In terms of the frequency $\nu = c/\lambda$, where c is the speed of light, this can be written as

$$I(\nu, T) = C_3\, \nu^3\, e^{-C_4 \nu/T}. \tag{2}$$

In January 1899, Planck [3] argued that this law should involve the characteristic energy kT, where k is Boltzmann's constant (first defined by Planck), and he used dimensional analysis to introduce the symbol b (later called h) such that $b\nu$ has dimensions of energy, and Wien's law can be rewritten as[1]

$$I(\nu, T) = C_5\, b\nu^3\, e^{-b\nu/kT}. \tag{3}$$

From the data on blackbody radiation available in 1899, Planck determined $b = h$ to within 4% of its presently accepted value. Thus, Planck introduced his constant h without any consideration of quantum physics. He considered that this constant was of fundamental significance, and in sec. 26 of [3], he used dimensional analysis to introduce the Planck length, mass, time and temperature.

As better data on blackbody radiation became available in 1899-1900, it became clear that Wien's distribution law was not accurate for long wavelengths, which led Planck to infer (October 1900, [5, 6]; see also [7]) that h represents the quantum of action, and that the oscillators of frequency ν in the walls of a blackbody cavity have energies that are integer multiples of $h\nu$.[2]

2. Closer to the theme of Ref. [1], we note that in 1912, Nicholson [10, 11, 12] (see also [13]) considered a precursor to the Bohr atom, involving a positive nucleus surrounded by n electrons equally spaced around a circular orbit of radius r, all with the same angular velocity ω. Nicholson supposed that there was a spectral line with this angular frequency (in contrast to Bohr's view [14] that spectral lines are due to transitions of electrons from one orbit to another).

[1] See the equation for u after eq. (53) of [3], where $a = b/k$. One of the few discussions of this in English is in [4]; see eq. (102) on p. 50.

[2] In 1911, Planck [8, 9] argued for the existence of zero-point energy $h\nu/2$ of an oscillator of frequency ν, such that possible energies are $(n + 1/2)\, h\nu$ for non-negative integer n. This could not be anticipated by dimensional analysis.

Nicholson supposed that the structure of an atom has something to do with Planck's constant h, whose dimensions are energy times time. So he declared (p. 679 of [12]) that the "energy" of the n electrons was nmv^2 (twice their kinetic energy), where m is the mass of an electron, whose velocity is $v = \omega r$. He took the orbital period, $2\pi/\omega$, to be the characteristic time of the atom, and assumed by dimensional analysis that $(nm\omega^2 r^2)(2\pi/\omega) = Nh$ for some integer N (not necessarily n). He also noted that the angular momentum of the n orbital electrons is $nmvr = nm\omega r^2 = Nh/2\pi = N\hbar$, and claimed that any change in the angular momentum of an atom involves an integer multiple of $\hbar$.

Following Nicholson, Bohr supposed (p. 25 of [14]) that the angular momentum of an atomic electron is an integer multiple of $\hbar = h/2\pi$, which was the key step towards his model of an atom. Bohr, unlike Nicholson, supposed that the spectral lines of an atom are due to transitions from an orbit of an excited electron to a lower-energy electron orbit.

Nicholson's effort illustrates that dimensional analysis alone is not sufficient to arrive at a good model of an atom.

References

[1] K. Glampedakis, *A dimensional analysis path to h and the Bohr atom structure*, Am. J. Phys. **94**, 346-350 (2026). https://doi.org/10.1119/5.0292670
https://kirkmcd.princeton.edu/examples/QM/glampedakis_ajp_94_346_26.pdf

[2] W. Wien, *On the Division of Energy in the Emission-Spectrum of a Black Body*, Phil. Mag. **43**, 214-220, (1897).
https://doi.org/10.1080/14786449708620983
http://kirkmcd.princeton.edu/examples/statmech/wien_pm_43_214_97.pdf

[3] M. Planck, *Über irreversible Strahlungsvorgänge*, Sitz. König. Preuss. Akad. Wissen. **26**, 440-480 (1899), http://kirkmcd.princeton.edu/examples/statmech/planck_skpaw_26_440_99.pdf
dated January 1899, and reprinted in Ann. d. Phys. **1**, 69-122 (1900).
https://doi.org/10.1002/andp.19003060105
http://kirkmcd.princeton.edu/examples/statmech/planck_ap_1_69_00.pdf

[4] O. Darrigol, *From c-Numbers to q-Numbers The Classical Analogy in the History of Quantum Theory* (U. Cal. Press, 1993).
https://publishing.cdlib.org/ucpressebooks/view?docId=ft4t1nb2gv
Click on the link to Chap. III in Part A.
See the bottom of p. 50 of Darrigol's book, where in eq. (102) Planck's a for Wien's distribution law (see the equation for u after eq. (53), p. 471, of his 1899 paper [3]) is now written as h/k where k is Boltzmann's constant (first defined by Planck!), and $b' = b/4\pi$, and b of Planck's 1899 paper was changed to h by him in 1900. Planck's version differs from Wien's version of 1896-97 only by dimensional analysis for the constants in the distribution law.

[5] M. Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum*, Verh. Deutsch. Phys. Gesell. **2**, 237-245 (1900).

http://kirkmcd.princeton.edu/examples/statmech/planck_vdpg_2_237_00.pdf
Translation: http://kirkmcd.princeton.edu/examples/QM/planck_vdpg_2_237_00_english.pdf

[6] M. Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum*, Ann. d. Phys. **4**, 553-563 (1901).
https://doi.org/10.1002/andp.19013090310
http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01.pdf
Translation: http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01_english.pdf

[7] M. Nauenberg, *Max Planck and the birth of the quantum hypothesis*, Am. J. Phys. **84**, 709-720 (2016). http://dx.doi.org/10.1119/1.4955146
http://kirkmcd.princeton.edu/examples/QM/nauenberg_ajp_84_709_16.pdf

[8] M. Planck, *Eine neue Strahlungshypothese*, Verh. Deutsch. Phys. Ges. **13**, 138-148 (1911).
https://www.google.com/books/edition/Verhandlungen_der_Deutschen_Physikalisch/D95aAAAAYAAJ?hl=en
http://kirkmcd.princeton.edu/examples/statmech/planck_vdpgb_13_138_11.pdf

[9] M. Planck, *Über die Begründung des Gesetzes der schwarzen Strahlung*, Ann. d. Phys. **37**, 642-656 (1912). https://doi.org/10.1002/andp.19123420403
http://kirkmcd.princeton.edu/examples/statmech/planck_ap_37_642_12.pdf

[10] J.W. Nicholson, *The Spectrum of Nebulium*, Mon. Not. Roy. Astro. Soc. **72**, 49-64 (1912). https://doi.org/10.1093/mnras/72.1.49
http://kirkmcd.princeton.edu/examples/QM/nicholson_mnras_72_49_11.pdf

[11] J.W. Nicholson, *The Constitution of the Solar Corona. I: Protofluorine*, Mon. Not. Roy. Astro. Soc. **72**, 139-150 (1912). https://doi.org/10.1093/mnras/72.2.139
http://kirkmcd.princeton.edu/examples/QM/nicholson_mnras_72_139_12.pdf

[12] J.W. Nicholson, *The Constitution of the Solar Corona. II*, Mon. Not. Roy. Astro. Soc. **72**, 677-693 (1912). https://doi.org/10.1093/mnras/72.8.677
http://kirkmcd.princeton.edu/examples/QM/nicholson_mnras_72_677_12.pdf

[13] R. McCormmach, *The Atomic Theory of John William Nicbolson*, Arch. Hist. Exact. Sci. **3**, 160-183 (1966). https://doi.org/10.1007/BF00357268
http://kirkmcd.princeton.edu/examples/QM/mccormmach_ahes_3_160_66.pdf

[14] N. Bohr, *On the Constitution of Atoms and Molecules*, Part 1. Phil. Mag. **26**, 1-23 (1913). https://doi.org/10.1080/14786441308634955
http://kirkmcd.princeton.edu/examples/QM/bohr_pm_1_1_13.pdf
Part II. Phil. Mag. **26**, 476-502 (1913). https://doi.org/10.1080/14786441308634993
http://kirkmcd.princeton.edu/examples/QM/bohr_pm_1_476_13.pdf
Part III. Phil. Mag. **26**, 857-875 (1913). https://doi.org/10.1080/14786441308635031
http://kirkmcd.princeton.edu/examples/QM/bohr_pm_1_857_13.pdf