Image Method for a Line Charge and a Dielectric Cylinder

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 8, 2026; updated October 31, 2026)

1 Introduction

It is claimed in two distinguished textbooks [1, 2] that an image method exists for a line charge parallel to the axis of an electrically neutral, linear dielectric cylinder, but there are small imperfections in the discussion there, which are reviewed in this note.[1]

2 What Should an Image Method Involve for Dielectric Media?

In static examples of electric charges in vacuum and (perfect) conductors, the image method considers the electric scalar potential outside the conductors to be due to the actual charges plus fictitious image charges inside the conductors. The electric charges not on the conductors can be called free charges, while those on the conductors can be called bound.

In static examples of electric charges and linear dielectric media, when conductors are not present, we consider free charges somehow embedded in the dielectric media. Such a free, point charge q induces polarization charges in the dielectric medium, say, of relative permittivity ϵ next to the free charge, In Gaussian units (as used through this note) the displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$ are related by $\mathbf{D} = \epsilon\mathbf{E}$, and close to the point charge we have $\mathbf{D} = q\,\hat{\mathbf{r}}_q/r_q^2$, and hence $\mathbf{E} = q\,\hat{\mathbf{r}}_q/\epsilon r_q^2$ close to the charge.

For a free, line charge q per unit length we have $\mathbf{D} = 2q\,\hat{\mathbf{r}}_q/r_q$, and hence $\mathbf{E} = 2q\,\hat{\mathbf{r}}_q/\epsilon r_q$ close to the line charge.

In electrostatics it is usual to take the vector potential $\mathbf{A}$ to be zero, such that the electric field can be computed from a scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla} V$, so the image method can be considered as a prescription for computing the potential V, from which the fields $\mathbf{E}$ and $\mathbf{D}$ can then be deduced.

Close to point charge q the potential V can be written as $V = q/\epsilon r_q$, and close to a line charge $V = -(2q/\epsilon)\ln r_q$, which behavior should be contained in the image method for dielectric media.

Furthermore, if the relative permittivity ϵ is large, the polarization charges move rather freely, and the medium approaches that of a conduitor, in which the static electric field is zero. This indicates that the potential in a dielectric medium of relative permittivity ϵ must include a factor of $1/\epsilon$

We now have a choice in the image method as to whether we consider the medium in which the image charges reside to be vacuum, or to be the same as the dielectric medium in

[1] A review of image methods is in Ref. [3]. There is no image method for a point charge and a dielectric sphere, as acknowledged in the footnote to Prob. 3 of §7 of Ref. [2].

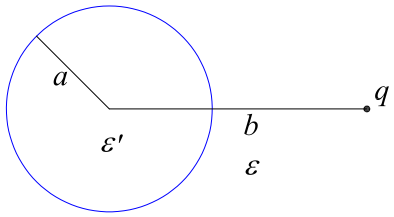
which a hypothetical observer resides. The better choice is the latter, which is used in the rest of this note.[2]

3 Landau's Analysis

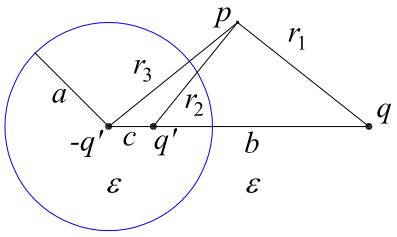
The argument of Landau on image methods for dielectric media, in Probs. 1- 4 of §7 of Ref. [2] is more compact than that of Smythe in sec. 4.04 of Ref. [1], so we review it first.[3]

Landau argued that the success of the image method for a (free) electric charge outside a conducting sphere suggests that the case two-dimensional of a (free) line charge q per unit length as distance b outside (or inside) a dielectric cylinder of radius a and relative permittivity ϵ', surrounded by an otherwise infinite medium of relativity permittivity ϵ, can be solved a version of the image method. While the equations in Probs. 1-4 of §7 of Ref. [2] indicate that he considered in the image method the medium in which the observer is located in infinite in extent, he did not state this verbally.

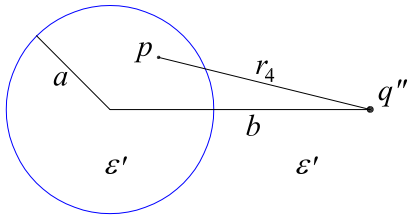
The figure below is for prob. 3 of Ref. [2], in which the line charge is outside the dielectric cylinder.



Landau found that for $r > a$ the potential V can be computed as that due to three line charges in an infinite medium of relative permittivity ϵ, with q at $(r, \theta) = (b, 0)$, line charge q' (per unit length) at $(r, \theta) = (c, 0)$ with $c = a^2/b$, and line charge $-q$ along the z-axis, as sketched in the figure below.



Observer at $r > a$



Observer at $r < a$

For $r < a$ the potential is due only to a line charge q'' (per unit length) at the position $(r, \theta) = (b, 0)$ of the actual line charge, but in an infinite medium of relative permittivity ϵ'.

In this argument there are only two unknowns, q' and q'', and there are two boundary conditions, the the radial component of $\mathbf{D}$, and the tangential component of $\mathbf{E}$, are continuous across the boundary at $r = a$. Hence, a solution is possible, which has the form

$$V(r > a) = -\frac{2q}{\epsilon} \ln r_1 - \frac{2q'}{\epsilon} \ln r_2 + \frac{2q'}{\epsilon} \ln r_3 \,, \qquad V(r < a) = -\frac{2q''}{\epsilon'} \ln r_4 \,, \tag{1}$$

[2]This prescription may have first been used by Searle (1898) [4] in an image method for permeable media.

[3]See also sec. 8.3.6 of [5] and Probs. 4.3 and 4.4 of [6].

with

$$c = \frac{a^2}{b}\,, \qquad q' = \frac{q(\epsilon - \epsilon')}{(\epsilon + \epsilon')}\,, \qquad q'' = \frac{2q\epsilon'}{(\epsilon + \epsilon')}\,, \tag{2}$$

4 Smythe's Analysis

In sec. 4.09 of Ref. [1], Smythe considered a line charge in vacuum outside a dielectric cylinder, and his eq. (7) of this section agrees with our eq. (1) for an image solution with $\epsilon = 1$ and $\epsilon' = K$ in Smythe's notation. A difficulty with his argument, based on Fourier analysis, is in eq. (6) of sec. 4.04, which contains a term in $\ln r$ in the Fourier analysis of the potential in the region $r < a$, which implies an infinite potential at the origin.

In the rest of this section we exhibit a more consistent Fourier analysis the potential.

Again we suppose the vector potential is everywhere zero in this electrostatic problem, such that the (static) electric field $\mathbf{E}$ can be deduced from an electric scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla} V$. Then, according to one of Maxwell's equations (in Gaussian units),

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho_{\text{total}} = -\nabla^2 V, \tag{3}$$

often called Poisson's equation, where ρ_{total} is the volume density of electric charge. In regions where $\rho_{\text{total}} = 0$ we have Laplace's equation,

$$\nabla^2 V = 0, \tag{4}$$

which admits solutions via the method of separation of variables.

In the present example ρ_{total} is nonzero only at the line charge at $(r, \theta) = (b, 0)$, and on the surface of the dielectric cylinder. This indicates that we can look for solutions via the method of separation of variables in the three regions, $0 < r < a$, $a < r < b$ and $r > b$.

In the present example the scalar potential has the symmetry $V(r, -\theta) = V(r, \theta)$, so a Fourier expansion for the potential (see, for example, eqs. (10)-(11) of sec. 4.01 of Ref. [1]) contains terms in $\cos k_n\theta$, but not $\sin k_n\theta$ for integer $n > 0$, nor in θ. Since in the present example $0 \le \theta \le 2\pi$ we must have $\cos 2\pi k_n = 1$, so $k_n = n$. Then, the potential V in the present example has the general form

$$V(r, \theta) = K + C \ln r + \sum_{n=1} \cos n\theta \left(A_n r^n + \frac{B_n}{r^n} \right), \tag{5}$$

for constants K, C, A_n and B_n.

In the first region, $0 \le r < a$, where $\rho_{\text{total}} = 0$, the potential must be finite, and so C and B_n are zero here.[4] In the third region, $r > b$, the potential can be at most logarithmically divergent, so A_n is zero here. Hence, we can write the potential for the present example as

$$V(r, \theta) = \begin{cases} K + \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (0 < r < a), \\ K' + C' \ln r + \sum_{n=1} \left(A'_n \left(\frac{r}{a}\right)^n + B'_n \left(\frac{a}{r}\right)^n \right) \cos n\theta & (a < r < b), \\ K'' + C'' \ln r + \sum_{n=1} B''_n \left(\frac{a}{r}\right)^n \cos n\theta & (r > b). \end{cases} \tag{6}$$

[4] This requirement is not satisfied by eq. (6) of sec. 4.04 of Ref. [1], so most of the analysis in that section is not valid.

For large r the $\mathbf{D}$ field, due to the free charge on the wire, is $\mathbf{D} \approx 2q\,\hat{\mathbf{r}}/r$, and so the electric field (a large r) is $\mathbf{E} = \mathbf{D}/\epsilon \approx 2q\,\hat{\mathbf{r}}/\epsilon r$. Hence the potential at large r is $-(2q/\epsilon)\ln r$, and so

$$C'' = -2q/\epsilon. \tag{7}$$

Continuity of the potential at $r = a$ and $r = b$ implies that $K + C\ln a = K' + C'\ln a$ and $K' + C'\ln b = K'' + C''\ln b$, independent of angle θ. This can be satisfied by taking[5]

$$K' = 0 = K'', \qquad C' = C'', \qquad \text{and} \qquad K = C''\ln a = -(2q/\epsilon)\ln a. \tag{8}$$

Continuity of the potential at $r = a$ also implies that

$$A_n = A'_n + B'_n. \tag{9}$$

A second relation among the constants of eq. (9) can be obtained by noting that the radial component of the electric displacement field $\mathbf{D} = \epsilon\mathbf{E} = -\epsilon\boldsymbol{\nabla}V$ is continuous across the boundary $r = a$ at which there can be bound charge but not free charge. Hence,

$$\epsilon' \frac{\partial V(a^-)}{\partial r} = \epsilon \frac{\partial V(a^+)}{\partial r}\,, \tag{10}$$

$$\epsilon' A_n = \frac{\epsilon}{\epsilon'}(A'_n - B'_n) = A_n + B'_n, \tag{11}$$

recalling eq. (9). Thus,

$$B'_n = A'_n \frac{\epsilon - \epsilon'}{\epsilon + \epsilon'}\,. \tag{12}$$

Using eq. (12) in eq. (9), we find

$$A'_n = A_n \frac{\epsilon + \epsilon'}{2\epsilon}\,, \tag{13}$$

and then eq. (12) also tells us that

$$B'_n = A_n \frac{\epsilon - \epsilon'}{2\epsilon}\,. \tag{14}$$

Continuity of the potential at $r = b$ tells us that

$$B''_n = A'_n \left(\frac{b}{a}\right)^{2n} + B'_n = A_n \left[\left(\frac{b}{a}\right)^{2n} \frac{\epsilon + \epsilon'}{2\epsilon} + \frac{\epsilon - \epsilon'}{2\epsilon}\right]. \tag{15}$$

The potential (6) now can be written as

$$V(r,\theta) = \begin{cases} -\frac{2q}{\epsilon}\ln a + \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (0 < r < a), \\ -\frac{2q}{\epsilon}\ln r + \sum_{n=1} A_n \left[\frac{\epsilon+\epsilon'}{2\epsilon}\left(\frac{r}{a}\right)^n + \frac{\epsilon-\epsilon'}{2\epsilon}\left(\frac{a}{r}\right)^n\right]\cos n\theta & (a < r < b), \\ -\frac{2q}{\epsilon}\ln r + \sum_{n=1} A_n \left[\frac{\epsilon+\epsilon'}{2\epsilon}\left(\frac{b^2}{ar}\right)^n + \frac{\epsilon-\epsilon'}{2\epsilon}\left(\frac{a}{r}\right)^n\right]\cos n\theta & (r > b). \end{cases} \tag{16}$$

[5]These choices will make our eventual solution (20) for the potential reduce to that of a line charge q (per unit length) at $(r,\theta) = (b,0)$ in vacuum, yielding eq. (21) below.

Finally, the remaining Fourier coefficients A_n can be determined from the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\rm free}$ by considering a Gaussian surface (of unit length in z) that surrounds the cylindrical shell (b,θ),

$$4\pi q_{\rm free,in} = \int \mathbf{D}\cdot d\mathbf{Area} = b\epsilon \int d\theta\ (E_{r^+} - E_{r^-}). \tag{17}$$

From this we learn that,

$$\begin{aligned} 4\pi q\delta(\theta) &= b\epsilon(E_{r^+} - E_{r^-}) = b\epsilon\left(-\frac{\partial V(b^+)}{\partial r} + \frac{\partial V(b^-)}{\partial r}\right) \\ &= (\epsilon+\epsilon')\sum_n nA_n\left(\frac{b}{a}\right)^n \cos n\theta. \end{aligned} \tag{18}$$

Multiplying eq. (18) by $\cos n\theta$ and integrating over θ we find that,

$$A_n = \frac{4q}{n(\epsilon+\epsilon')}\left(\frac{a}{b}\right)^n. \tag{19}$$

With this, the potential (16) becomes

$$V(r,\theta) = \begin{cases} -\frac{2q}{\epsilon}\ln a + \frac{4q}{\epsilon+\epsilon'}\sum_{n=1}\frac{1}{n}\left(\frac{r}{b}\right)^n\cos n\theta & (0<r<a), \\ -\frac{2q}{\epsilon}\ln r + \frac{4q}{\epsilon+\epsilon'}\sum_{n=1}\frac{1}{n}\left[\frac{\epsilon+\epsilon'}{2\epsilon}\left(\frac{r}{b}\right)^n + \frac{\epsilon-\epsilon'}{2\epsilon'}\left(\frac{a^2/b}{r}\right)^n\right]\cos n\theta & (a<r<b), \\ -\frac{2q}{\epsilon}\ln r + \frac{4q}{\epsilon+\epsilon'}\sum_{n=1}\frac{1}{n}\left[\frac{\epsilon+\epsilon'}{2\epsilon}\left(\frac{b}{r}\right)^n + \frac{\epsilon-\epsilon'}{2\epsilon}\left(\frac{a^2/b}{r}\right)^n\right]\cos n\theta & (r>b). \end{cases} \tag{20}$$

To interpret this as due to a set of line charges, we note that for an infinite dielectric medium of relative permittivity ϵ', eq. (20) indicates that the potential for a single (free) line charge q (per unit length) at $(r,\theta) = (b,0)$ would be

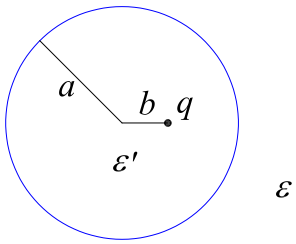
$$V_{q\ \text{in}\ \epsilon'}(r,\theta) = \begin{cases} -\frac{2q}{\epsilon'}\ln b + \frac{2q}{\epsilon'}\sum_{n=1}\frac{1}{n}\left(\frac{r}{b}\right)^n\cos n\theta & (0<r<b), \\ -\frac{2q}{\epsilon'}\ln r + \frac{2q}{\epsilon'}\sum_{n=1}\frac{1}{n}\left(\frac{b}{r}\right)^n\cos n\theta & (r>b). \end{cases} \tag{21}$$

We now identify the potential of eq. (20) for $r<a$, inside the dielectric cylinder, as that (to within an additive constant) due to a line charge $2q\epsilon'/(\epsilon+\epsilon')$ (per unit length) at the location of the actual line charge at $(r,\theta) = (b,0)$ in the medium for $r<a$ with relative permittivity ϵ' (*or line charge* $2q/(\epsilon+\epsilon')$ *per unit length in vacuum*).

For the region $r>a$ we would like to identify the potential as that of a line charge q at the position of the actual line charge in the medium with relative permittivity ϵ, together with a line charge $q' = q(\epsilon-\epsilon')/\epsilon(\epsilon+\epsilon')$ (per unit length) at $(r,\theta) = (a^2/b,0)$ inside the cylinder and line charge $-q'$ along the z-axis. Supposing in the image method that for observers with $r>a$ the media have relative permittivity ϵ everywhere the potentials for all three lines charges should include a factor of $1/\epsilon$. This is consistent with eq. (21) when we add and subtract $2q\ln r(\epsilon-\epsilon')/\epsilon(\epsilon+\epsilon')$ to the forms for $r>a$ in eq. (20). Then, the result of the Fourier analysis agrees with eqs. (1)-(2) of Landau's analysis.

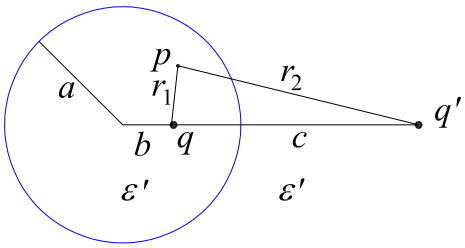
5 Line Charge inside the Dielectric Cylinder

An image solution also exists for a free line charge q per unit length at distance $b < a$ from the axis of a dielectric cylinder of radius a with relative permittivity ϵ' surrounded by a medium of relative permittivity ϵ, as sketched in the figure below.

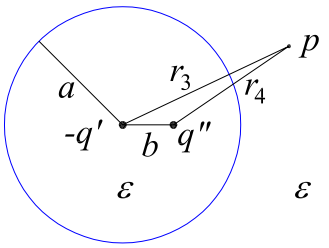


The image solution for the potential at $r < a$ involves line charge q at $r = b$ and line charge $q' = q(\epsilon - \epsilon')/(\epsilon + \epsilon')$ at $c = a^2/b$, both in vacuum, as sketched in the left figure below.

The potential at $r > a$ involves line charge $q'' = 2q/(\epsilon + \epsilon')$ at $r = b$ and line charge $-q'$ along the axis, as in the right figure below.



Observer at $r < a$



Observer at $r > a$

The potential has the form,

$$V(r < a) = -\frac{2q}{\epsilon'} \ln r_1 - \frac{2q'}{\epsilon'} \ln r_2 \,, \qquad V(r > a) = \frac{2q'}{\epsilon} \ln r_3 - \frac{2q''}{\epsilon} \ln r_4 \,. \tag{22}$$

This case is discussed in prob. 4, §7 of Ref. [2], and prob. 4.4 of Ref. [6].

This note was inspired by e-discussions with Elly England.

References

[1] W.R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, 1950).
https://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf
https://kirkmcd.princeton.edu/examples/EM/smythe_68.pdf

[2] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media.* (Pergamon, 1960). 1st Russian edition 1959.
https://kirkmcd.princeton.edu/examples/EM/landau_ecm2.pdf

[3] K.T. McDonald, *Dielectric (and Magnetic) Image Methods* (Nov. 21, 2009).
https://kirkmcd.princeton.edu/examples/i,age.pdf

[4] G.F.C. Searle, *On the Magnetic Field due to a Current in a Wire Placed Parallel to the Axis of a Cylinder of Iron*, Electrician **40**, 453 (1898).
https://kirkmcdprinceton.edu/examples/EM/searle_electrician_40_453_98.pdf

[5] A. Zangwill, *Modern Electrodynamics* (Cambridge U. Press, 2012),
https://kirkmcd.princeton.edu/examples/EM/zangwill_em_12.pdf

[6] F. Lacava, *Classical Electrodynamics*, 2nd ed. (Springer, 2022).
https://kirkmcd.princeton.edu/examples/EM/lacava_22.pdf