

Image Method for a Line Charge and a Dielectric Cylinder

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

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1 Introduction

It is claimed in two distinguished textbooks [1, 2] that an image method exists for a line charge parallel to the axis of an electrically neutral, linear dielectric cylinder, but there are small imperfections in the discussion there, which are reviewed in this note.¹

2 What Should an Image Method Involve for Dielectric Media?

For stationary charges and conductors in vacuum, the image method is a prescription for computing the electric field $\mathbf{E}$ (and the force $q\mathbf{E}$ on a charge q) using only a set of charges in vacuum. Here, all charges except bound charges in the interior of conductors can be called “free” charges. When (linear) dielectric media are present we must also consider the field $\mathbf{D} = \epsilon\mathbf{E}$ where, in Gaussian units as used in this note, ϵ is the relative permittivity of a dielectric medium. Computation of the force on a free charge q_{free} in a dielectric medium is complicated by the presence of polarization charges in the dielectric medium next to the charge. We are less interested in the force on q_{free} than that on q_{total} as needed to hold the free charge at rest in, say, a gaseous medium.

The image method should give a prescription for computing $\mathbf{E}$ in the dielectric media along with an awareness of the relation between q_{free} and q_{total} . In electrostatics it is usual to take the vector potential $\mathbf{A}$ to be zero, such that the electric field can be computed from a scalar potential V according to $\mathbf{E} = -\nabla V$, so the image method can be considered as a prescription for computing the potential V , and hence $\mathbf{E}$, in dielectric media based on charges in vacuum.

As for the relation between q_{free} and q_{total} we note that the electric field near a “point” charge is dominated by the self field of the charge. For a line charge q_{free} (per unit length) in a medium of relative permittivity ϵ , the nearby $\mathbf{D}$ field would be $2q_{\text{free}}\hat{\mathbf{r}}_q/r_q$ for r_q measured with respect to the line charge. Then, close to the charge $\mathbf{E} = \mathbf{D}/\epsilon = 2q_{\text{free}}\hat{\mathbf{r}}_q/\epsilon r_q$, which is due to a total line charge (per unit length)

$$q_{\text{total}} = \frac{q_{\text{free}}}{\epsilon}. \quad (1)$$

The image method associated with a free charge q in a dielectric medium of relative permittivity ϵ should involve image charge q/ϵ , rather than q , at the position of the actual

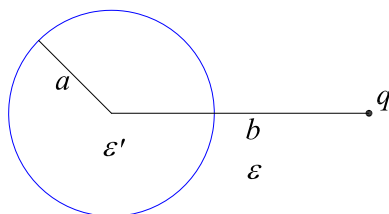
¹A review of image methods is in Ref. [3]. There is no image method for a point charge and a dielectric sphere, as acknowledged in the footnote to Prob. 3 of §7 of Ref. [2].

charge in a computation based only on charges in vacuum.²

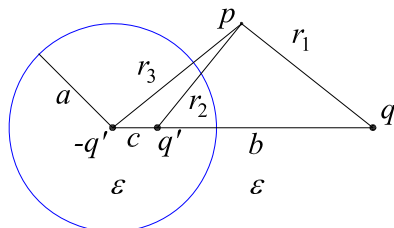
3 A Delicacy in Landau's Analysis

The argument of Landau, in Probs. 3 and 4 of §7 of Ref. [2] is more compact than that of Smythe in sec. 4.04 of Ref. [1], so we review it first.³

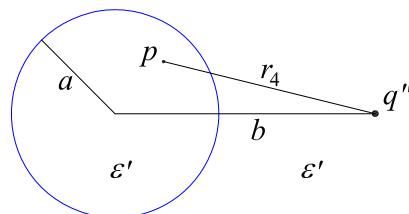
Landau argued that the success of the image method for a (free) electric charge outside a conducting sphere suggests that the case two-dimensional of a (free) line charge q per unit length at distance b outside (or inside) a dielectric cylinder of radius a and relative permittivity ϵ' , surrounded by an otherwise infinite medium of relative permittivity ϵ , can be solved a version of the image method. The figure below supposes the line charge is outside the dielectric cylinder



Landau found that for $r > a$ the potential V can be computed as that due to three line charges in an infinite medium of relative permittivity ϵ , with q at $(r, \theta) = (b, 0)$, line charge q' (per unit length) at $(r, \theta) = (c, 0)$ with $c = a^2/b$, and line charge $-q$ along the z -axis, as sketched in the figure below.



Observer at $r > a$



Observer at $r < a$

For $r < a$ the potential is due only to a line charge q'' (per unit length) at the position $(r, \theta) = (b, 0)$ of the actual line charge, but in an infinite medium of relative permittivity ϵ' .

In this argument there are only two unknowns, q' and q'' , and there are two boundary conditions, the the radial component of $\mathbf{D}$, and the tangential component of $\mathbf{E}$, are continuous across the boundary at $r = a$. Hence, a solution is possible, which has the form

$$V(r > a) = \frac{q}{\epsilon r_1} + \frac{q'}{\epsilon r_2} - \frac{q'}{\epsilon r_3}, \quad V(r, a) = \frac{q''}{\epsilon' r_4}, \quad (2)$$

with

$$c = \frac{a^2}{b}, \quad q' = \frac{q(\epsilon - \epsilon')}{(\epsilon + \epsilon')}, \quad q'' = \frac{2q\epsilon}{(\epsilon + \epsilon')}, \quad (3)$$

²Many authors prefer an image method involving only a single, infinite dielectric medium, which leads to confusion in examples where two different dielectric media are involved.

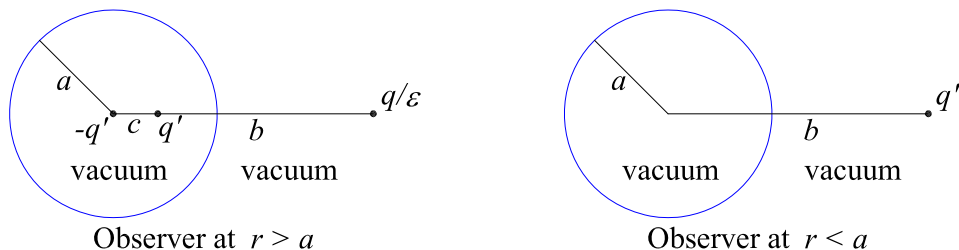
³A version of Landau's argument is given in Probs. 4.3 and 4.4 of [4]. See also sec. 8.3.6 of [5].

If instead we suppose the image method should consist of charges in vacuum, as in the figure below, the form of the potential would be

$$V(r > a) = \frac{q/\epsilon}{r_1} + \frac{q'}{r_2} - \frac{q'}{r_3}, \quad V(r, a) = \frac{q''}{r_4}, \quad (4)$$

with

$$c = \frac{a^2}{b}, \quad q' = \frac{q(\epsilon - \epsilon')}{\epsilon(\epsilon + \epsilon')}, \quad q'' = \frac{2q\epsilon}{\epsilon'(\epsilon + \epsilon')}, \quad (5)$$



4 A Difficulty with Smythe's Analysis

In sec. 4.09 of Ref. [1], Smythe considered a line charge in vacuum outside a dielectric cylinder, and his eq. (7) of this section agrees with our eq. (2) for an image solution with $\epsilon = 1$ and $\epsilon' = K$ in Smythe's notation. A difficulty with his argument, based on Fourier analysis, is in eq. (6) of sec. 4.04, which contains a term in $\ln r$ in the Fourier analysis of the potential in the region $r < a$, which implies an infinite potential at the origin.

In the rest of this section we exhibit a more consistent Fourier analysis the potential.

Again we suppose the vector potential is everywhere zero in this electrostatic problem, such that the (static) electric field $\mathbf{E}$ can be deduced from an electric scalar potential V according to $\mathbf{E} = -\nabla V$. Then, according to one of Maxwell's equations (in Gaussian units),

$$\nabla \cdot \mathbf{E} = 4\pi\rho_{\text{total}} = -\nabla^2 V, \quad (6)$$

often called Poisson's equation, where ρ_{total} is the volume density of electric charge. In regions where $\rho_{\text{total}} = 0$ we have Laplace's equation,

$$\nabla^2 V = 0, \quad (7)$$

which admits solutions via the method of separation of variables.

In the present example ρ_{total} is nonzero only at the line charge at $(r, \theta) = (b, 0)$, and on the surface of the dielectric cylinder. This indicates that we can look for solutions via the method of separation of variables in the three regions, $0 < r < a$, $a < r < b$ and $r > b$.

In the present example the scalar potential has the symmetry $V(r, -\theta) = V(r, \theta)$, so a Fourier expansion for the potential (see, for example, eqs. (10)-(11) of sec. 4.01 of Ref. [1]) contains terms in $\cos k_n \theta$, but not $\sin k_n \theta$ for integer $n > 0$, nor in θ . Since in the present

example $0 \leq \theta \leq 2\pi$ we must have $\cos 2\pi k_n = 1$, so $k_n = n$. Then, the potential V in the present example has the general form

$$V(r, \theta) = K + C \ln r + \sum_{n=1} \cos n\theta \left(A_n r^n + \frac{B_n}{r^n} \right), \quad (8)$$

for constants K , C , A_n and B_n .

In the first region, $0 \leq r < a$, where $\rho_{\text{total}} = 0$, the potential must be finite, and so C and B_n are zero here.⁴ In the third region, $r > b$, the potential can be at most logarithmically divergent, so A_n is zero here. Hence, we can write the potential for the present example as

$$V(r, \theta) = \begin{cases} K + \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (0 < r < a), \\ K' + C' \ln r + \sum_{n=1} \left(A'_n \left(\frac{r}{a}\right)^n + B'_n \left(\frac{a}{r}\right)^n \right) \cos n\theta & (a < r < b), \\ K'' + C'' \ln r + \sum_{n=1} B''_n \left(\frac{a}{r}\right)^n \cos n\theta & (r > b). \end{cases} \quad (9)$$

For large r the $\mathbf{D}$ field, due to the free charge on the wire, is $\mathbf{D} \approx 2q \hat{\mathbf{r}}/r$, and so the electric field (a large r) is $\mathbf{E} = \mathbf{D}/\epsilon \approx 2q \hat{\mathbf{r}}/\epsilon r$. Hence the potential at large r is $-(2q/\epsilon) \ln r$, and so

$$C'' = -2q/\epsilon. \quad (10)$$

Continuity of the potential at $r = a$ and $r = b$ implies that $K + C \ln a = K' + C' \ln a$ and $K' + C' \ln b = K'' + C'' \ln b$, independent of angle θ . This can be satisfied by taking⁵

$$K' = 0 = K'', \quad C' = C'', \quad \text{and} \quad K = C'' \ln a = -(2q/\epsilon) \ln a. \quad (11)$$

Continuity of the potential at $r = a$ also implies that

$$A_n = A'_n + B'_n. \quad (12)$$

A second relation among the constants of eq. (12) can be obtained by noting that the radial component of the electric displacement field $\mathbf{D} = \epsilon \mathbf{E} = -\epsilon \nabla V$ is continuous across the boundary $r = a$ at which there can be bound charge but not free charge. Hence,

$$\epsilon' \frac{\partial V(a^-)}{\partial r} = \epsilon \frac{\partial V(a^+)}{\partial r}, \quad (13)$$

$$\epsilon' A_n = \frac{\epsilon}{\epsilon'} (A'_n - B'_n) = A_n + B'_n, \quad (14)$$

recalling eq. (12). Thus,

$$B'_n = A'_n \frac{\epsilon - \epsilon'}{\epsilon + \epsilon'}. \quad (15)$$

⁴This requirement is not satisfied by eq. (6) of sec. 4.04 of Ref. [1], so most of the analysis in that section is not valid.

⁵These choices will make our eventual solution (23) for the potential reduce to that of a line charge q (per unit length) at $(r, \theta) = (b, 0)$ in vacuum, yielding eq. (24) below.

Using eq. (15) in eq. (12), we find

$$A'_n = A_n \frac{\epsilon + \epsilon'}{2\epsilon}, \quad (16)$$

and then eq. (15) also tells us that

$$B'_n = A_n \frac{\epsilon - \epsilon'}{2\epsilon}. \quad (17)$$

Continuity of the potential at $r = b$ tells us that

$$B''_n = A'_n \left(\frac{b}{a}\right)^{2n} + B'_n = A_n \left[\left(\frac{b}{a}\right)^{2n} \frac{\epsilon + \epsilon'}{2\epsilon} + \frac{\epsilon - \epsilon'}{2\epsilon} \right]. \quad (18)$$

The potential (9) now can be written as

$$V(r, \theta) = \begin{cases} -\frac{2q}{\epsilon} \ln a + \sum_{n=1} A_n \left(\frac{r}{a}\right)^n \cos n\theta & (0 < r < a), \\ -\frac{2q}{\epsilon} \ln r + \sum_{n=1} A_n \left[\frac{\epsilon + \epsilon'}{2\epsilon} \left(\frac{r}{a}\right)^n + \frac{\epsilon - \epsilon'}{2\epsilon} \left(\frac{a}{r}\right)^n \right] \cos n\theta & (a < r < b), \\ -\frac{2q}{\epsilon} \ln r + \sum_{n=1} A_n \left[\frac{\epsilon + \epsilon'}{2\epsilon} \left(\frac{b^2}{ar}\right)^n + \frac{\epsilon - \epsilon'}{2\epsilon} \left(\frac{a}{r}\right)^n \right] \cos n\theta & (r > b). \end{cases} \quad (19)$$

Finally, the remaining Fourier coefficients A_n can be determined from the Maxwell equation $\nabla \cdot \mathbf{D} = 4\pi\rho_{\text{free}}$ by considering a Gaussian surface (of unit length in z) that surrounds the cylindrical shell (b, θ) ,

$$4\pi q_{\text{free, in}} = \int \mathbf{D} \cdot d\mathbf{Area} = b\epsilon \int d\theta (E_{r+} - E_{r-}). \quad (20)$$

From this we learn that,

$$\begin{aligned} 4\pi q\delta(\theta) &= b\epsilon(E_{r+} - E_{r-}) = b\epsilon \left(-\frac{\partial V(b^+)}{\partial r} + \frac{\partial V(b^-)}{\partial r} \right) \\ &= (\epsilon + \epsilon') \sum_n n A_n \left(\frac{b}{a}\right)^n \cos n\theta. \end{aligned} \quad (21)$$

Multiplying eq. (21) by $\cos n\theta$ and integrating over θ we find that,

$$A_n = \frac{4q}{n(\epsilon + \epsilon')} \left(\frac{a}{b}\right)^n. \quad (22)$$

With this, the potential (19) becomes

$$V(r, \theta) = \begin{cases} -\frac{2q}{\epsilon} \ln a + \frac{4q}{\epsilon + \epsilon'} \sum_{n=1} \frac{1}{n} \left(\frac{r}{b}\right)^n \cos n\theta & (0 < r < a), \\ -\frac{2q}{\epsilon} \ln r + \frac{4q}{\epsilon + \epsilon'} \sum_{n=1} \frac{1}{n} \left[\frac{\epsilon + \epsilon'}{2\epsilon} \left(\frac{r}{b}\right)^n + \frac{\epsilon - \epsilon'}{2\epsilon'} \left(\frac{a^2/b}{r}\right)^n \right] \cos n\theta & (a < r < b), \\ -\frac{2q}{\epsilon} \ln r + \frac{4q}{\epsilon + \epsilon'} \sum_{n=1} \frac{1}{n} \left[\frac{\epsilon + \epsilon'}{2\epsilon} \left(\frac{b}{r}\right)^n + \frac{\epsilon - \epsilon'}{2\epsilon} \left(\frac{a^2/b}{r}\right)^n \right] \cos n\theta & (r > b). \end{cases} \quad (23)$$

To interpret this as due to a set of line charges in vacuum, for which $\epsilon = 1 = \epsilon'$, we can use these values in eq. (23) to obtain the potential for a single line charge q (per unit length) in vacuum at $(r, \theta) = (b, 0)$ as

$$V_{q \text{ in vacuum}}(r, \theta) = \begin{cases} -2q \ln b + 2q \sum_{n=1} \frac{1}{n} \left(\frac{r}{b}\right)^n \cos n\theta & (0 < r < b), \\ -2q \ln r + 2q \sum_{n=1} \frac{1}{n} \left(\frac{b}{r}\right)^n \cos n\theta & (r > b). \end{cases} \quad (24)$$

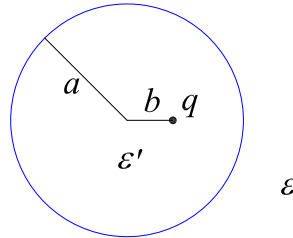
We now identify the potential of eq. (23) for $r < a$, inside the dielectric cylinder, as that (to within an additive constant) due to a line charge $2q/(\epsilon + \epsilon')$ (per unit length) in vacuum at the location of the actual line charge at $(r, \theta) = (b, 0)$.

For the region $r > a$ we would like to identify the potential as that of a line charge q/ϵ' (per unit length) in vacuum at the position of the actual line charge, together with a line charge $q' = q(\epsilon - \epsilon')/\epsilon(\epsilon + \epsilon')$ (per unit length) at $(r, \theta) = (a^2/b, 0)$ inside the cylinder and line charge $-q$ along the z -axis, with all three line charges in vacuum. This is consistent with eq. (24) when we add and subtract $2q \ln r(\epsilon - \epsilon')/\epsilon(\epsilon + \epsilon')$ to the forms for $r > a$ in eq. (23).

Then, the result of the Fourier analysis, interpreted as an image method, agrees with the corrected version (2) of Landau's analysis.

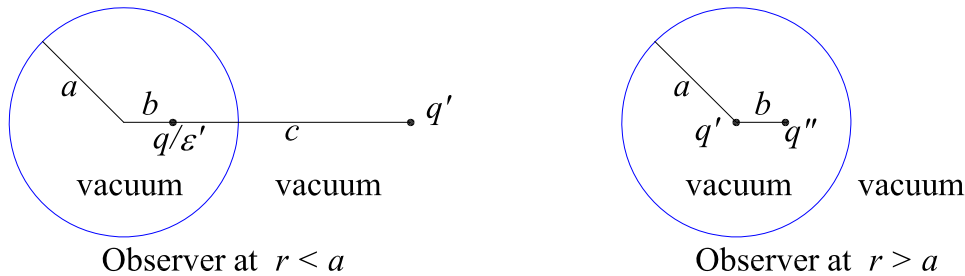
5 Line Charge inside the Dielectric Cylinder

An image solution also exists for a free line charge q per unit length at distance $b < a$ from the axis of a dielectric cylinder of radius a with relative permittivity ϵ' surrounded by a medium of relative permittivity ϵ , as sketched in the figure below.



The image solution for the potential at $r < a$ involves line charge q/ϵ' at $r = b$ and line charge $q' = q(\epsilon - \epsilon')/\epsilon'(\epsilon + \epsilon')$ at $c = a^2/b$, both in vacuum, as sketched in the left figure below.

The potential at $r > a$ involves line charge $q'' = 2q/(\epsilon + \epsilon')$ at $r = b$ and line charge q' along the axis, both in vacuum, as in the right figure below.



This case is discussed in prob. 4, §7 of Ref. [2], and prob. 4.4 of Ref. [4] where their e and λ should be e/ϵ' and λ/ϵ' .

This note was inspired by e-discussions with Elly England.

References

- [1] W.R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, 1950).
https://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf
https://kirkmcd.princeton.edu/examples/EM/smythe_68.pdf
- [2] L.D. Landau, E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*. (Pergamon, 1960). 1st Russian edition 1959.
https://kirkmcd.princeton.edu/examples/EM/landau_ecm2.pdf
- [3] K.T. McDonald, *Dielectric (and Magnetic) Image Methods* (Nov. 21, 2009).
https://kirkmcd.princeton.edu/examples/i_age.pdf
- [4] F. Lacava, *Classical Electrodynamics*, 2nd ed. Springer, 2022).
https://kirkmcd.princeton.edu/examples/EM/lacava_22.pdf
- [5] A. Zangwill, *Modern Electrodynamics* (Cambridge U. Press, 2012),
https://kirkmcd.princeton.edu/examples/EM/zangwill_em_12.pdf