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§ 1. Introduction.

In a recent paper* (cited henceforth as I) I have shown that the phenomena concerning the emission or reflexion of electrons by metals can be treated by calculating the emission or reflexion coefficient for the electrons at the surface of the metal and integrating over all incident electrons according to the electron theory of conductivity of Sommerfeld. In a further paper† (cited henceforth as II), R. H. Fowler and the present author have treated the cold emission in intense electric fields on the same principle.

The surface of the metal is characterised thereby as a region with a very sudden variation of the potential, that, according to the wave mechanics causes a reflexion. The emission coefficient is, of course, the ratio of the number of electrons going through to the number of incident electrons, and the relation

\[ R + D = 1 \]  

(R = reflexion coefficient, D = emission coefficient) is therefore always valid.

In the papers mentioned above, R and D have been calculated for the idealised forms of the potential steps \( \alpha \) and \( \eta \) of fig. 1, C denoting the total height of the potential step.

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The real form of the potential is, of course, a smooth curve, and it is the purpose of this paper to give a more accurate calculation of D and R for a field that approaches much closer to the real circumstances.

The result for the intense field effect is a small decrease of the necessary field strengths, whereas the emission coefficient for the thermions, which was estimated in I at about \( \frac{1}{2} \), becomes very nearly equal to 1, so that the average reflexion in that case amounts only to a few per cent.

§ 2. The Field and the Differential Equation Employed.

The chief deviation of the real fields from those of fig. 1 consists in the image field, which must give the right kind of rounding off of the upper corner of the potential curves of fig. 1.* For our purposes, furthermore, the form of the upper part is of much greater importance than the lower part, since according to I (formula (32)) the expression, which measures the reflexion in a given interval, contains \((W - U)^{-3/2}\) as factor.

\( W \) denotes here the energy of the normal component of the velocity in the interior of the metal, the other components being of no importance, and \( U \) the potential energy. Therefore the regions, where \( W - U \) is small, give a much larger effect, whereas at great velocities of the electrons irregularities of the potential do not matter so much.

We have, therefore, quite a good approximation for our purpose by taking the image potential right down to the bottom of the total potential step, and connecting it there with the constant potential in the interior of the metal.

The above consideration justifies also the treating of the problem as the refraction of the de Broglie waves in a continuous medium, since the wavelength, \( \lambda = \frac{\hbar}{mv} \), is large compared with the atomic distances for the critical region with small \( W - U \).

Now the image potential is equal to \(-\frac{e^2}{4\lambda}x\), where \( x \) is the distance from the metal surface, and we take as the total potential therefore

\[
U = C - \frac{e^2}{4\lambda}x - Fx \quad \text{for} \quad x > x_0 \tag{2a}
\]
\[
U = 0 \quad \text{for} \quad x < x_0 \tag{2b}
\]

where \( x_0 \) is given by

\[
\frac{e^2}{4\lambda x_0} = C, \tag{3}
\]

* An experimental proof of this is furnished by the good agreement of the Schottky correction (Schottky, 'Z. Physik,' vol. 14, p. 63 (1923)) for the influence of not too strong fields on the thermionic emission, which has been verified by various investigators. See, for instance, Pforte, 'Z. Physik,' vol. 49, p. 333 (1928), and de Bruyne, 'Roy. Soc. Proc.,' A, vol. 120, p. 423 (1928).
and we have thus the following graphs in the cases $F = V$ (fig. 2a) and $F \neq V$ (fig. 2b).

The value of $C$ itself, i.e., the total potential difference between the inside and the outside of the metal, can be estimated quite well as the sum of the partial potential

$$\mu = \frac{\hbar^2}{8m} \left( \frac{6n}{\pi g} \right)^{2/3}$$

(4)

where $n$ is the number of free electrons per cubic centimetre, and $g = 2$ the statistical weight of quantum state of an electron, and the thermionic work function $\chi$. This has been checked quite well by observing the refractive index for the de Broglie waves in the experiments about their Laue diffraction.*

Now for one ionised electron per atom $\mu$ amounts to already about 8 volts and for two electrons per atom to 12 volts, and $C = \mu + \chi$ may vary therefore between 10 and 20 volts for various metals.

We consider at first only the case $F = 0$, i.e., without an external field. We have then to solve the wave equations

$$\frac{d^2 \psi}{dx^2} + \kappa^2 \left( W - C + \frac{e^2}{4} \right) \psi = 0 \quad \text{for } x > x_0, \quad (4a)$$

$$\frac{d^2 \psi}{dx^2} + \kappa^2 W \psi = 0 \quad \text{for } x < x_0, \quad (4b)$$

under the condition that both, $\psi$ and $d\psi/dx$ are continuous at $x = x_0$ and $\psi$ represents asymptotically for $x \gg x_0$ a stream of outgoing electrons.

The problem is therefore to find that special solution of (4a) for large $x$ and to obtain a development of it in the neighbourhood of $x_0$, which is small.

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* See, for instance, the summary given by Rosenfeld and Witmer, 'Z. Physik,' vol. 49, p. 534 (1923).
This can be done by the integral of Barnes,* which allows us to connect the solutions for large and small $x$.

§ 3. The Solution of the Differential Equation.

By the substitution

$$x = \frac{1}{2}z (W - C)^{-1}/\kappa,$$

(5)

the differential equation (4A) becomes

$$\frac{d^2\psi}{dz^2} + (1 + \frac{1}{2}k)\psi = 0,$$

(6)

where

$$k = \frac{1}{2}k e^2 (W - C)^{-\frac{1}{2}}; \quad z_0 = e^2\kappa (W - C)^{\frac{1}{2}}/2C.$$

(7)

The magnitude in powers of which the final development will proceed is

$$s = z_0 k = e^2 e^4/16C,$$

(8)

which is for $C = 2 \cdot 10^{-11}$ (approximately 12 volts) equal to 0.25.

Now putting

$$\psi = e^{-iz} z^{-ik} v,$$

(9)

we get as equation for $v$

$$z^2 \frac{d^2 v}{dz^2} - 2ikz \frac{dv}{dz} + ik (ik + 1) v - iz^2 \frac{dv}{dz} = 0,$$

or changing the variable

$$z = -it,$$

(10)

$$t^2 \frac{d^2 v}{dt^2} - 2ikt \frac{dv}{dt} + ik (ik + 1) v - t^2 \frac{dv}{dt} = 0.$$

(11)

Of course a second solution of (6) is obtained by taking a positive sign in the exponential function in (9), but the above solution represents the outgoing waves that we need.

Equation (11) is now satisfied by the integral

$$I = \int_{-\infty}^{+\infty} \Gamma(s) \Gamma(-s + ik) \Gamma(-s + ik + 1) t^s ds.$$

(12)

* For a full development of this method see Whittaker and Watson, 'Modern Analysis,' Cambridge, 4 ed., 1927, p. 243; also all other references in respect of analytical matters will be given to this treatise.
The poles of the integrand and the path of integration are indicated in fig 3, and the validity of (12) is shown as follows

\[ \int_{-\infty}^{\infty} \Gamma(i) \Gamma(-s+ik) \Gamma(-s+ik+1) \{ \Gamma(s+1) - \Gamma(-s+ik+1) \} ds, \]

which is equal to

\[ \left( \int_{-\infty}^{\infty} - \int_{1}^{\infty} \right) \Gamma(s+1) \Gamma(-s+ik+1) \Gamma(-s+ik+2) t^s ds. \]

Since there are no poles of the last integrand between the paths of integration, and the integrand tends to zero as \(|s| \to \infty\), the last expression vanishes according to Cauchy's theorem, and (12) is, therefore, a solution of (11). The behaviour of the above integrands for large \(s\) can thereby be estimated by the asymptotic expression of the gamma function (cf. Whittaker and Watson, p. 278).

\[ \Gamma(s+a) \to s^{a-\frac{1}{2}} e^{-s} (1). \]

The integrand in (12) is accordingly for large \(|s|\) of the order of magnitude

\[ |s|^{-s+2k-\frac{1}{2}} e^{t^s}. \]

It tends to zero for all large \(s\) with positive real part, but finite \(t\), and one can therefore transform the contour of integration into small circles around the poles on the right side, and it is thus

\[ I = -2\pi i \sum \text{Residues at } s = ik, \quad ik + 1, \quad ik + 2, \ldots. \]

This gives a series for \(v\) that converges for all finite values of \(t\).
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On the other hand for every \( s \) on the left side of the \( i \)-axis a \( t \) can be found so that the integrand in (12) becomes as small as desired, and this can be used to obtain an asymptotic expansion for large \( t \) by using a transformation of the contour of integration as indicated in fig. 4. This is, of course, the analytical

\[
\text{Fig. 4.}
\]

continuation of the solution for finite \( t \), and allows one thus to study the behaviour of the latter near infinity.

The asymptotic solution has the form

\[
I = 2\pi i \sum \text{Residues at } s = 0, -1, -2, \ldots r
\]

plus a remainder \( R_r \), which can be made as small as desired for large enough \( k \)'s and as

\[
\text{Res } \Gamma(z) = \frac{(-1)^n}{n!}, \quad (n = \text{positive integer})
\]

we obtain

\[
v = I = 2\pi i \sum_0^r \left( -\frac{1}{n} \right) t^{-n} \Gamma(n + ik) \Gamma(n + ik + 1) + R_r
\]

\[
= 2\pi i (ik) \Gamma(ik + 1) \sum_0^r (-1)^n t^{-n} [ik(ik + 1)] \ldots [(ik + n - 1)(ik + n)]/n! + R_r, \quad (14)
\]

i.e., a single expansion in descending powers of \( t \), which shows that (9) really represents outgoing waves.

The expansion for finite values of \( t \) becomes, always omitting irrelevant constant factors,

\[
v = \sum \text{Res } F
\]

where

\[
F = \Gamma(s) \Gamma(-s + ik) \Gamma(-s + ik + 1) t'. \quad (16)
\]
The pole \( s = \imath k \) is a simple one and has the residue
\[
\text{Res } F = \frac{\Gamma (i \kappa)}{\Gamma (ik)} t^i k.
\]
(17)

The poles \( s = i k + 1, \ i k + 2, \ldots \) are of the order two.

With the help of the relation (cf. Whittaker and Watson, p. 239)
\[
\Gamma (z) \Gamma (1 - z) = \pi / \sin \pi z,
\]
\( F \) can be written
\[
F = - \pi^2 g / \sin^2 \pi (s - i k) ; \quad g = \Gamma (s) t^s / \Gamma (1 + s - i k) \Gamma (s - i k)
\]
(18)

where \( g \) is regular for the values of \( s \) considered. Now we have the development in rational fractions
\[
\pi^2 / \sin^2 \pi z = \sum_{n=0}^{+\infty} (z - n)^{-2},
\]
and the residue of \( F \) becomes therefore
\[
\text{Res } F = \left[ - \frac{dg}{ds} \right]_{s=i(k+n)}.
\]
(19)

From (15), (17), (18) and (19) we obtain (omitting the factor \(-\Gamma (i k)\))
\[
v = t^i k + \sum_{n=1}^{\infty} \left[ \frac{d}{ds} \Gamma (s) t^s / \Gamma (i k) \Gamma (1 + s - i k) \Gamma (s - i k) \right]_{s=i(k+n)}
\]
\[
= t^i k \left( 1 + \sum_{n=1}^{\infty} \frac{d}{dz} \Gamma (z + i k) t^z / \Gamma (i k) \Gamma (1+z) \Gamma (z) \right)_{z=n}
\]
(20)

By repeated application of the formula
\[
\frac{\Gamma' (z + a)}{\Gamma (z + a)} = \frac{1}{z + a - 1} + \frac{\Gamma' (z + a - 1)}{\Gamma (z + a - 1)},
\]
which is obtained by logarithmic differentiation of
\[
\Gamma (z + a) = (z + a - 1) \Gamma (z + a - 1),
\]
one gets
\[
\left[ \frac{d}{dz} \left( \frac{1}{\Gamma (z)} \right) \right]_{z=n} = - \frac{1}{\Gamma (n)} \left( -\frac{1}{n+1} + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \right); \quad n = 1, 2, \ldots
\]
(21A)

\[
\left[ \frac{d}{dz} \left( \frac{1}{\Gamma (z+1)} \right) \right]_{z=n} = - \frac{1}{\Gamma (n+1)} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \right),
\]
(21B)

\[
[\Gamma' (z + i k)]_{z=n} = \Gamma (n + i k) \left( \frac{1}{i k} + \frac{1}{i k + 1} + \ldots + \frac{1}{i k + n - 1} + \frac{\Gamma' (i k)}{\Gamma (i k)} \right),
\]
(21C)

\[\gamma = \Gamma' (z) / \Gamma (z) \text{ for } z = 1 \text{ being Euler's constant (} = 0.5772 \ldots\).}
Carrying out the differentiation in (20) with the help of these formulæ one obtains

\[ v = t^{ik} \left[ 1 + \sum_{n=1}^{\infty} t^n A_n (\log t + B_n^*) \right], \tag{22} \]

where

\[ A_n = \Gamma(n+ik)/\Gamma(ik) \Gamma(n+1) \Gamma(n) = ik(ik+1) \ldots (ik+n-1)/n!(n+1)! \] \tag{23A}

\[ B_n^* = 2\gamma + \frac{1}{n} - 2 \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) + \frac{1}{ik} + \frac{1}{ik+1} + \cdots \frac{1}{ik+n-1} + \frac{\Gamma'(ik)}{\Gamma(ik)}. \tag{23B} \]

Returning to the original variable \( z = -it \) (cf. (10)), we obtain from (9) (omitting again \( i^{ik} \) in the second row) as the final result

\[ \psi = e^{-iz} z^{-ik} v(iz) \]

\[ = e^{-iz} \left[ 1 + \sum_{n=1}^{\infty} i^{zn} A_n (\log z + B_n) \right], \tag{24} \]

where

\[ B_n = B_n^* + \log i = B_n^* + \frac{1}{2}i\pi. \tag{25} \]

It may be remarked that

\[ \psi = e^{-iz} \sum_{n=1}^{\infty} i^{zn} A_n \tag{26} \]

is a second independent solution of (6), which may be obtained by the ordinary series method, but, of course, does not represent outgoing waves. The method used above offers, incidentally, also quite a convenient way of discussing the solutions of the Schrödinger equation for the hydrogen atom.

To complete our solution we have only to obtain a suitable expression for \( \Gamma'(ik)/\Gamma(ik) \). Since, according to (7), \( k \) is becoming very large in the neighbourhood of \( W = C \), we need an expansion for large \( k \). As it is not allowed to differentiate asymptotic expansions directly, we have to proceed in the following way. From the general formula

\[ \Gamma'(z)/\Gamma(z) = -\gamma - \frac{1}{2} + z \sum_{n=1}^{\infty} n^{-1} (z+n)^{-1}, \]

(Whittaker and Watson, p. 241) we obtain

\[ \Gamma'(ik)/\Gamma(ik) = -\gamma + i/k + \psi + i\phi, \tag{27} \]

where

\[ \psi = k^2 \sum_{n=1}^{\infty} (k^2 + n^2)^{-1}; \quad \phi = k \sum_{n=1}^{\infty} (k^2 + n^2)^{-1}. \tag{28} \]
These series can be summed by Euler's formula (compare Whittaker and Watson, p. 128)

\[ f(n) + f(n+1) + \ldots f(m) = \int_n^m f(x) \, dx + \frac{f(m) + f(n)}{2} + \sum_{r=1}^{s} (-1)^{r-1} \frac{B_r}{(2r)!} \left( f^{(2r-1)}_{(m)} - f^{(2r-1)}_{(n)} \right) + R_s \]  

(29)

where \( n \) and \( m \) are here positive integers and \( m > n \); \( B_r \) are the Bernoulli numbers \( (B_1 = 1/6) \), and the remainder is of the order of magnitude

\[ R_s = \mathcal{O} \left( \int_n^m f^{(2s+1)}(x) \, dx \right). \]

Taking \( n = 1, m = \infty \); \( f(x) = k^2/x (k^2 + x^2) \) we obtain

\[ \psi = \int_1^\infty \frac{dx}{x (1 + x^2/k^2)} + \frac{k^2}{2 (k^2 + 1)} + \frac{1}{12} \left[ \frac{1}{x^2} \left( \frac{1 + x^2/k^2 + 2x^2/k^2 + \cdots}{1 + x^2/k^2} \right) \right]_1^\infty + \ldots \]

\[ = \left[ \log z - \frac{1}{2} \log (1 + 2z) \right]_{1/k}^\infty + \frac{1}{2} \frac{1}{2k^2} + \frac{1}{12} \frac{k^2 + 3}{k^2 + 2} + \frac{1}{12} \frac{k^2 + 2 + 1/k^2}{k^2 + 2} + \ldots \]

(30)

Similarly \( \phi \) becomes

\[ \phi = \int_1^\infty \frac{dx}{k (1 + x^2/k^2)} + \frac{1}{2 (k + 1/k)} + \ldots \]

\[ = \left[ \arctan z \right]_{1/k}^\infty + \frac{1}{2k} + \ldots = \frac{1}{2k} + 0 \left( \frac{1}{k} \right), \]

(31)

and therefore the required asymptotic expression is

\[ \frac{\Gamma'(ik)}{\Gamma(ik)} = \log k - \gamma + \frac{1}{2} + i \left( \frac{1}{2} \pi + \frac{1}{2k} \right) + \ldots, \]

(32)

which gives according to (23b) and (25)

\[ B_n = + \gamma + \frac{7}{12} + \frac{1}{n} - 2 \left( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} \right) + \log k + \frac{1}{ik} \]

\[ + \frac{1}{ik + 1} + \ldots \frac{1}{ik + n - 1} + i \left( \frac{\pi + 1}{2k} \right) + \ldots \]

(33)

§ 4. Calculation of the Emission or Reflection Coefficient.

Taking as solution for (4b)

\[ \psi = ae^{-ik (x-x_0) \sqrt{w}} + a' e^{+ik (x-x_0) \sqrt{w}}, \]

(34)
the continuity conditions for $\psi$ and $d\psi/dx$ at $x_0$ are

$$a + a' = \psi \left( x_0 \right)$$

$$i k \left( -a + a' \right) W^z = \left[ \frac{d\psi}{dx} \frac{dz}{dx} \right]_{x_0} = 2k \left( W - C \right)^\dagger \left[ \frac{d\psi}{dx} \right]_{x_0}.$$

Now, since according to (24)

$$\frac{d\psi}{dz} = -\frac{i}{z} e^{-i\pi/2} \left[ 1 + \sum_{n=1}^\infty \left( iz \right)^n A_n (\log z + B_n) - 2 \sum_{n=1}^\infty \left( iz \right)^n A_n (1 + n \log z + n B_n) \right],$$

(35)

the above relations can be written in the form

$$a + a' = S$$

$$-a + a' = -\rho T,$$

(36)

where

$$S = 1 + \sum_{n=1}^\infty \left( iz \right)^n A_n (\log z + B_n)$$

$$T = S - 2 \sum_{n=1}^\infty \left( iz \right)^n A_n (1 + n \log z + n B_n)$$

(37)

$$\rho = \left( (W - C)/W \right)^\dagger,$$

(38)

and the emission coefficient becomes

$$D \left( W \right) = 1 - \frac{\left| a' \right|^2}{\left| a \right|^2} = \frac{2\rho \left( ST + \bar{ST} \right)}{\left| S \right|^2 + \rho^2 \left| T \right|^2 + \rho \left( ST + \bar{ST} \right)}.$$ 

(39)

The evaluation of this expression with the help of (23a), (33) and (37) is rather tiresome but quite elementary. All terms can be expressed as functions of

$$X = \left( (W - C)/C \right) \text{ and } s = z_0 k = k^2 e^4/16C,$$

(40)

the second of which is for $C = 6 \times 10^{-12}$ equal to $0.25$, and is the only constant that depends on the special metal. For discussing the results it is convenient to expand numerator and denominator in (39) in powers of $X$, the coefficients of which are itself series in $S$. Neglecting higher powers than $X^2$ and $S^2$ we obtain as the final result

$$D \left( W \right) = \frac{a + b X + c X^2 + \ldots}{d + e X + f X^2 + \ldots} = \alpha + \beta X + \gamma X^2 + \ldots,$$

(41)

where

$$a = 4\pi \sqrt{s} \; ; \; b = 4 \cdot 16 \cdot 49 \cdot s^2 \; ; \; c = -2\pi \sqrt{s} \left( 1 + 4s \right)$$

$$d = 1 + s \left[ (\log s)^2 + 0.311 \log s + 10.90 \right]$$

$$+ s^2 \left[ - (\log s)^2 + 0.689 \log s - 9.23 \right]$$

$$e = -6.28 \sqrt{s} \left( 1 + s \right) + \frac{1}{2} (a + b)$$

$$f = 2 - s \left[ (\log s)^2 + 8 \cdot 321 \log s + 14 \cdot 18 \right]$$

$$+ s^2 \left[ - (\log s)^2 + 1 \cdot 96 \log s + 9 \cdot 23 \right] + \frac{1}{2} (a + b)$$

and

$$\alpha = a/d \; ; \; \beta = (bd - ea)/d^2 \; ; \; \gamma = (cd^2 - efd - fad + e^2 a)/d^3.$$ 

(42)
The influence on the thermionic emission is given by the average emission coefficient (compare I (17)).

\[
\overline{D} = \frac{1}{kT} \int_{C}^{\infty} D(W) e^{-(W-C)/kT} dW = \int_{0}^{\infty} D(kTz) e^{-z} dz
\]

\[
= \int_{0}^{\infty} (\alpha + \beta \sqrt{kT/C} \sqrt{z + \gamma kT/C} + ...) e^{-z} dz
\]

\[
= \alpha + 0.886 \beta \sqrt{kT/C} + \gamma kT/C + ... \quad (44)
\]

The expansions (41) or (44) are, of course, valid only for sufficiently small \(X = (W - C)/C\) or \(kT/C\), where \(W - C\) is the energy of the escaping electrons and \(C\) the total amount of the potential difference on the surface.

For the special value \(C = 2 \cdot 10^{-11}\), \(\text{i.e., } s = 0.24\), we get the following numerical values,

\[
\alpha = 0.927 \quad \beta = 0.431 \quad \gamma = -1.018.
\]

The emission coefficient does not tend therefore to zero for vanishing escaping velocity, but only to the already very high value 0.927. At the same time the reflection coefficient \(R = 1 - D\) for electrons, hitting the surface from the outside, is only about 0.07 for the slowest electrons, and it seems, therefore, hardly possible to observe this effect.

Furthermore it follows from this calculation that the coefficient \(A\) in the Richardson-Dushman formula for thermionic emission

\[
i = AT^2 e^{-b/T},
\]

which is according to the theory (compare I (18))

\[
A = 2\pi me^2 g\overline{D}/k^3 \quad (46)
\]

has a value near 120 and not 60 amp./cm.², which is obtained by neglecting the factor \(g\overline{D}\) (\(g = 2\) being the weight factor for the electronic quantum states due to the spin). But it seems exceedingly difficult, owing to the overwhelming influence of the exponential function, to determine \(A\) with sufficient accuracy to decide between those values experimentally.

The difference between the results of this more accurate determination of \(D\) and \(R\) and my former estimation of it is, of course, due to the very slow vanishing of the image field for large distances, which causes the emission coefficient to approach a non-zero limit instead of a limit zero as at a sharp potential step.
§ 5. The Case of Intense Electric Fields.

We consider now the case of an external field. The rigorous treatment of it would demand a complete discussion of the differential equation (compare 4A)

$$\frac{d^2\psi}{dx^2} + k^2 \left( W - C + \frac{e^2}{4x} + Fx \right) \psi = 0,$$
(47)

which has not yet been done in the mathematical literature, as far as I know. (47) is, of course, a special case of the equation for the Stark effect, as (4A) is for the hydrogen atom, but existing investigations do not give what we want here, and it seems to be very difficult to obtain a complete solution. But for values of W that are rather smaller than C, a method of taking account of the image field has already been suggested in II, § 4. The most important part of the equation for emission in strong electric fields (II (18) and (21)) is the exponential factor, which we determined, using the field of fig. 1 (ii) to be

$$e^{-2Q} = \exp \left[ -4\kappa (C - W)^{3/2}/3F \right].$$
(48)

Now, applying the method of Jeffreys (compare I, § 4), one finds that for an arbitrary potential of a shape like that of fig. 5, the exponent becomes

$$2Q = 2\kappa \int_{x_1}^{x_2} (U - W)^3 dx,$$
(49)

the integration being taken over the shaded area in fig. 5. For our potential (2A) we have therefore

$$2Q = 2\kappa \int_{x_1}^{x_2} (C - W - e^2/4x - Fx)^3 dx,$$
(50)
\[ x_1 \text{ and } x_2 \text{ being the zero points of the integrand, and, therefore, putting } C - W = A \]
\[ x_{2.1} = \left\{ A \pm \sqrt{(A^2 - e^2 F)} \right\}/2F. \] (51)

The integral in (50) is a simple elliptic integral that can be transformed in the usual way to the following normal form
\[ 2Q = \frac{2 \sqrt{2}}{3F} A \sqrt{\left\{ A^2 + \sqrt{(A^2 - e^2 F)} \right\}} \left[ E(k) - \frac{F(e^2)}{A \left\{ A + \sqrt{(A^2 - e^2 F)} \right\}} K(k) \right], \] (52)
where \( E \) and \( K \) are the elliptic normal integrals *
\[ K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)^{-1} d\phi, \quad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi) d\phi, \]
and
\[ k = 2 \sqrt{(A^2 - e^2 F)}/(A + \sqrt{(A^2 - e^2 F)}). \] (53)

The emission formula will become to a sufficient approximation (compare II (21))
\[ I = \text{const } F^2 e^{-3Q}, \] (54)
where in the formula (52) for \( Q \) one has to take the thermionic work function \( \chi \) as the value for \( A \). According to this formula, of course, \( \log I \) will not give exactly a straight line when plotted against \( 1/F \), but the deviation is so small that it will most probably not be possible to observe it.

For the discussion and comparison of (54) with our former calculation it may be noted, that \( e \sqrt{F} = \Delta C \) is the drop in the work function effected by the external field, as pointed out first by Schottky. The quotient \( V \) of the new value of \( Q \) (52) and the former one (compare (43) is)
\[ V = \sqrt{\left(1 + \sqrt{(1 - x^2)}\right)/2} \left( E(k) - \frac{x^2}{1 + \sqrt{(1 - x^2)}} K(k)\right). \] (55)
\[ x = e \sqrt{F}/\chi; \quad k = 2 \sqrt{1 - x^2/(1 + \sqrt{1 - x^2})}, \]
\( V \) that measures directly the influence of the image force on the cold electronic emission, depends only on \( x \), i.e., the ratio of the Schottky correction to the undisturbed work function, and the values of it are given by the following table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V )</td>
<td>1</td>
<td>0.951</td>
<td>0.904</td>
<td>0.849</td>
<td>0.781</td>
<td>0.696</td>
<td>0.603</td>
<td>0.494</td>
<td>0.345</td>
<td>0</td>
</tr>
</tbody>
</table>

* Tables of the elliptic integrals are given, for instance, in 'Zahnke-Emde, Funktionen-tafeln,' Leipzig, 1909, p. 68.
and the emission formula can be written with sufficient approximation

\[ I = \text{const} \; e^{-4kx^2v/vF}. \]  

(56)

If we measure \( \chi \) in volts and \( F \) in volts per centimetre we obtain

\[ x = 3.78 \times 10^{-4} \sqrt{F/\chi}. \]

For \( \chi = 4.5 \) volts, which corresponds to pure tungsten, and \( F = 3 \times 10^7 \), at which figure the emission becomes sensible (according to our formula, II (22)), \( x \) becomes 0.46, and the \( v \) therefore about 0.8, which means a corresponding reduction for the necessary field strengths. That reduction, of course, is far from sufficient to bring the coefficient of \( 1/F \) down to the observed values and one has still to assume surface irregularities or sensitive spots.

The transition from the cold emission to the thermionic emission could be treated rigorously only by solving equation (47) for values of \( W \) that are a little smaller but very nearly equal to \( C \). But it appears quite clearly from the present calculations that the two effects will be fairly well additive. Since nearly all electrons with a larger energy than the potential threshold can escape, it is to be expected that the Schottky correction for thermionic emission holds quite well in conformity with the experimental results. On the other hand the cold emission is caused by the much larger number of slower electrons according to the Sommerfeld distribution function of the electrons in the metal that can escape only under the influence of the external field.

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