Skiing on a Cosine Hill

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1 Problem

In a variant of the famous problem of skiing/sliding on a cylindrical hill, consider a hill with surface \( y = y_0 + A \cos(kx) \) (perhaps formed by glaciers). What is the largest value of \( k \) (for a given \( A \)) such that a skier who starts from rest at the top of the hill never leaves the (frictionless) surface while sliding down?

2 Solution

When the skier has traveled horizontal distance \( x \) from the top of the frictionless hill his/her speed is given by,

\[
v^2 = 2g\Delta y = 2gA(1 - \cos kx),
\]

where \( g \) is the acceleration due to gravity. At that position, the hill has angle \( \theta > 0 \) to the horizontal given by,

\[
\tan \theta = |y'| = kA |\sin kx|,
\]

\[
\sin \theta = \frac{|y'|}{1 + y'^2} = \frac{kA |\sin kx|}{1 + k^2A^2 \sin^2 kx},
\]

and radius of curvature \( R \),

\[
R = \frac{(1 + y'^2)^{3/2}}{|y''|} = \frac{(1 + k^2A^2 \sin^2 kx)^{3/2}}{k^2A |\cos kx|}.
\]

When \( \cos kx > 0 \) the center of curvature is below the surface, and the normal component of Newton’s equation of motion is,

\[
m g \sin \theta - N = \frac{mv^2}{R} \quad (\cos kx > 0),
\]

where \( N \) is the normal force of the surface on the skier. If the skier loses contact with the hill at angle \( \theta \), then \( N = 0 \) and,

\[
\sin \theta = \frac{kA \sin kx}{1 + k^2A^2 \sin^2 kx} = \frac{v^2}{gR} = 2A(1 - \cos kx) \frac{k^2A \cos kx}{(1 + k^2A^2 \sin^2 kx)^{3/2}},
\]

\[
\sin kx(1 + k^2A^2 \sin^2 kx)^{1/2} = 2kA(1 - \cos kx) \cos kx,
\]

\[
\sin^2 kx(1 + k^2A^2 \sin^2 kx) = 4k^2A^2(1 - 2 \cos kx + \cos^2 kx) \cos^2 kx,
\]
\[(1 - \cos^2 kx)[1 + k^2 A^2(1 - \cos^2 kx)] = 4k^2 A^2(\cos^2 kx - 2\cos^3 kx + \cos^4 kx), \quad (8)\]

\[f(\cos kx) = 3k^2 A^2 \cos^4 kx - 8k^2 A^2 \cos^3 kx + (1 + 6k^2 A^2) \cos^2 kx - 1 - k^2 A^2 = 0. \quad (9)\]

We are interested in the special case that the quartic polynomial \(f\) barely has a real solution \(x_0\), which implies that this solution is also at the minimum of \(f\),

\[
0 = \frac{df(\cos kx_0)}{d \cos kx} = 12k^2 A^2 \cos^3 kx_0 - 24k^2 A^2 \cos^2 kx_0 + 2(1 + 6k^2 A^2) \cos kx_0
= 2 \cos kx_0 (6k^2 A^2 \cos^2 kx_0 - 12k^2 A^2 \cos kx_0 + 1 + 6k^2 A^2). \quad (10)
\]

The solution \(\cos kx_0 = 0\) to eq. (10) is not a solution to the quartic equation (8), so if a minimum exists with \(f = 0\) it must be that,

\[
\cos^2 kx_0 - 2 \cos kx_0 + 1 + \frac{1}{6k^2 A^2} = 0. \quad (11)
\]

However, eq. (11) has no real solution, so we conclude that the skier never loses contact with a cosine hill for any values of \(A\) and \(k\).

The radius of curvature (3) increases with \(x\) up to \(x = \pi/2k\) where it is infinite, so the cosine hill is gentler than a cylindrical hill of radius \(r = R(0) = 1/k^2 A\), and is sufficiently gentle that the skier never loses contact with a cosine hill.