Comay's Paradox: Do Magnetic Charges Conserve Energy?

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1 Problem

The interaction energy of an electric charge q and a "point" electric dipole with moment $\mathbf{p}$ of fixed magnitude, both at rest, is, in Gaussian units,

$$\begin{aligned} U_{\rm int} &= \int \frac{\mathbf{E_p}\cdot\mathbf{E}_q}{4\pi}\,d\mathrm{Vol} = -\frac{1}{4\pi}\int \boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)\cdot\mathbf{E}_q\,d\mathrm{Vol} \\ &= -\frac{1}{4\pi}\int \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\mathbf{E}_q\right)d\mathrm{Vol} + \frac{1}{4\pi}\int \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\boldsymbol{\nabla}\cdot\mathbf{E}_q\,d\mathrm{Vol} \\ &= -\frac{1}{4\pi}\int_{\rm S\ at\ \infty}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\mathbf{E}_q\cdot d\mathbf{Area} + \int\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}q\,\delta^3(\mathbf{r}-\mathbf{r}_q)\,d\mathrm{Vol} = \mathbf{p}\cdot\frac{q\mathbf{r}_q}{r_q^3} \\ &= -\mathbf{p}\cdot\mathbf{E}_q \qquad \text{(electric charge and permanent electric dipole)}, \end{aligned} \tag{1}$$

taking the electric dipole to be at the origin, such that the field at the dipole (*i.e.*, at the origin) due to charge q at location $\mathbf{r}_q$ is $\mathbf{E}_q = -q\mathbf{r}_q/r_q^3$ with $\boldsymbol{\nabla}\cdot\mathbf{E}_q = 4\pi q\,\delta^3(\mathbf{r}-\mathbf{r}_q)$, while the field of the electric dipole can be related to the gradient of a scalar potential as $\mathbf{E_p} = -\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{r}/r^3)$. The force on the electric dipole $\mathbf{p}$ is,

$$\mathbf{F_p} = (\mathbf{p}\cdot\boldsymbol{\nabla}_\mathbf{p})\mathbf{E}_q = \boldsymbol{\nabla}_\mathbf{p}(\mathbf{p}\cdot\mathbf{E}_q) = -\boldsymbol{\nabla}_\mathbf{p}U_{\rm int}, \tag{2}$$

where $\boldsymbol{\nabla}_\mathbf{p}$ is the gradient at the location of the dipole $\mathbf{p}$, and we note that $\boldsymbol{\nabla}\times\mathbf{E}_q = 0$ for charge q at rest. The force on the electric charge q is,

$$\mathbf{F}_q = q\,\mathbf{E_p} = -q\,\boldsymbol{\nabla}_q\frac{\mathbf{p}\cdot\mathbf{r}_q}{r_q^3} = \boldsymbol{\nabla}_q(\mathbf{p}\cdot\mathbf{E}_q) = -\boldsymbol{\nabla}_q U_{\rm int} = \boldsymbol{\nabla}_\mathbf{p}U_{\rm int} = -\mathbf{F}_{\rm p}, \tag{3}$$

where $\boldsymbol{\nabla}_q$ is the gradient at the location of the charge q.

Similarly, interaction energy of a (Gilbertian) magnetic charge p (aka magnetic monopole) and a "point" magnetic dipole $\mathbf{m}_{\rm G}$ consisting of a pair of opposite magnetic charges with fixed separation, all at rest, is,

$$\begin{aligned} U_{\rm int} &= \int\frac{\mathbf{B}_{\mathbf{m}_{\rm G}}\cdot\mathbf{B}_p}{4\pi}\,d\mathrm{Vol} \\ &= -\mathbf{m}_{\rm G}\cdot\mathbf{B}_p \qquad \text{(Gilbertian magnetic charge and Gilbertian magnetic dipole)}, \end{aligned} \tag{4}$$

taking the magnetic dipole to be at the origin, such that the field at the dipole (*i.e.*, at the origin) due to magnetic charge p at location $\mathbf{r}_p$ is $\mathbf{B}_p = -p\,\mathbf{r}_p/r_p^3$ with $\boldsymbol{\nabla}\cdot\mathbf{B}_p = 4\pi p\,\delta^3(\mathbf{r}-\mathbf{r}_p)$, while the field of the magnetic dipole can be related to the gradient of a scalar potential as $\mathbf{B}_{\mathbf{m}_{\rm G}} = -\boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mathbf{r}/r^3)$. The force on the Gilbertian magnetic dipole $\mathbf{m}_{\rm G}$ is,

$$\mathbf{F}_{\mathbf{m}_{\rm G}} = (\mathbf{m}_{\rm G}\cdot\boldsymbol{\nabla}_\mathbf{m})\mathbf{B}_p = \boldsymbol{\nabla}_\mathbf{m}(\mathbf{m}_{\rm G}\cdot\mathbf{B}_p) = -\boldsymbol{\nabla}_\mathbf{m}U_{\rm int}, \tag{5}$$

where $\boldsymbol{\nabla}_{\mathbf{m}}$ is the gradient at the location of the dipole $\mathbf{m}$, and we note that $\boldsymbol{\nabla} \times \mathbf{B}_p = 0$ for magnetic charge p at rest. The force on the magnetic charge p (in vacuum) is,

$$\mathbf{F}_p = p\,\mathbf{B}_{\mathbf{m}_\mathrm{G}} = -p\,\boldsymbol{\nabla}_p \frac{\mathbf{m}_\mathrm{G} \cdot \mathbf{r}_p}{r_p^3} = \boldsymbol{\nabla}_p(\mathbf{m}_\mathrm{G} \cdot \mathbf{B}_p) = -\boldsymbol{\nabla}_p U_\mathrm{int} = \boldsymbol{\nabla}_{\mathbf{m}} U_\mathrm{int} = -\mathbf{F}_{\mathbf{m}_\mathrm{G}}, \tag{6}$$

where $\boldsymbol{\nabla}_p$ is the gradient at the location of the magnetic charge p.

However, if the "point" magnetic dipole $\mathbf{m}_\mathrm{A}$ is Ampèrian, such as that of an electron, proton or neutron, for which $\boldsymbol{\nabla} \cdot \mathbf{B}_{\mathbf{m}_\mathrm{A}} = 0$,[1] its interaction energy with a Gilbertian magnetic pole appears to vanish,

$$\begin{aligned}
U_\mathrm{int} &= \int \frac{\mathbf{B}_{\mathbf{m}_\mathrm{A}} \cdot \mathbf{B}_p}{4\pi}\, d\mathrm{Vol} = -\frac{1}{4\pi} \int \mathbf{B}_{\mathbf{m}_\mathrm{A}} \cdot \boldsymbol{\nabla} \frac{p}{r}\, d\mathrm{Vol} \\
&= -\frac{1}{4\pi} \int \boldsymbol{\nabla} \left(\frac{p}{r} \mathbf{B}_{\mathbf{m}_\mathrm{A}} \right) d\mathrm{Vol} + \frac{1}{4\pi} \int \frac{p}{r} \boldsymbol{\nabla} \cdot \mathbf{B}_{\mathbf{m}_\mathrm{A}}\, d\mathrm{Vol} \\
&= -\frac{1}{4\pi} \int_{\mathrm{S\ at\ } \infty} \frac{p}{r} \mathbf{B}_{\mathbf{m}_\mathrm{A}} \cdot d\mathbf{Area} \\
&= 0 \qquad \text{(Gilbertian magnetic charge and Ampèrian magnetic dipole)},
\end{aligned} \tag{7}$$

taking the magnetic charge to be at the origin, and noting that the field of a magnetic dipole falls off as $1/r^3$ at large distances.[2] The force on the Ampèrian magnetic dipole $\mathbf{m}_\mathrm{A}$ is,[3]

$$\mathbf{F}_{\mathbf{m}_\mathrm{A}} = \boldsymbol{\nabla}_{\mathbf{m}}(\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p), \tag{8}$$

while the force on the magnetic charge p (in vacuum) is,

$$\mathbf{F}_p = p\,\mathbf{B}_{\mathbf{m}_\mathrm{A}} = -p \boldsymbol{\nabla}_p \frac{\mathbf{m}_\mathrm{A} \cdot \mathbf{r}_p}{r_p^3} = \boldsymbol{\nabla}_p(\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p) = -\boldsymbol{\nabla}_{\mathbf{m}}(\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p) = -\mathbf{F}_{\mathbf{m}_\mathrm{A}}, \tag{9}$$

It is agreeable that $\mathbf{F}_p = -\mathbf{F}_{\mathbf{m}_\mathrm{A}}$, but neither of these forces is associated with a conserved field energy unless the interaction energy were $U_\mathrm{int} = -\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p$, rather than zero as found in eq. (7).

The implication is that the interaction of a Gilbertian magnetic charge with a "point" (permanent) Ampèrian magnetic dipole does not conserve energy.[4]

Can this be so?

This problem arose from considerations that classical electromagnetism might not be consistent if both electric and magnetic charges exist, as perhaps first discussed by Rohrlich [4], and also by Comay [5, 6, 7, 8, 11, 14], by Lipkin and Peshkin [9, 10], by Tejedor and Rubio [12], and by Getino, Rojo and Rubio [13]. The present example was first discussed by Comay in [5, 6], and further discussed in [9, 12, 13, 14].[5]

[1]For discussion of how we know that the magnetic moment of a neutron is Ampèrian, see [1].

[2]The argument of eq. (7) does not depend on the Ampèrian current loop being pointlike, but only that the magnetic field $\mathbf{B}_{\mathbf{m}_\mathrm{A}}$ of the loop obeys $\boldsymbol{\nabla} \cdot \mathbf{B}_{\mathbf{m}_\mathrm{A}} = 0$ and that this field falls off as $1/r^3$ at large distances.

[3]See, for example, sec. 5.7 of [2].

[4]The case of an Ampérian magnetic moment of finite size, whose steady current is maintained by a "battery", is considered in Appendix A.

[5]That magnetic monopoles might be problematic in quantum field theory is discussed in [15].

2 Solution

2.1 Delta Functions Associated with "Point" Dipoles

As discussed, for example, in sec. 4.1 of [2], the field of a "point" dipole $\mathbf{m}$ at the origin consisting of a pair of opposite (Gilbertian) charges can be written as,

$$\mathbf{B}_{\mathbf{m},\mathrm{G}} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3} - \frac{4\pi}{3}\mathbf{m}\,\delta^3(\mathbf{r}), \tag{10}$$

where the delta function at the origin describes the large field between the two charges that point from the positive to the negative charge, *i.e.*, opposite to the direction of the momentum $\mathbf{m}$. In contrast, as discussed in sec. 5.6 of [2], if the magnetic dipole is due to an (Ampèrian) loop of electric current, the field at the origin points in the same direction as the moment $\mathbf{m}$, with,

$$\mathbf{B}_{\mathbf{m},\mathrm{A}} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\,\delta^3(\mathbf{r}) = \mathbf{B}_{\mathbf{m},\mathrm{G}} + 4\pi\mathbf{m}\,\delta^3(\mathbf{r}). \tag{11}$$

Hence,

$$\begin{aligned}\int \frac{\mathbf{B}_{\mathbf{m},\mathrm{Amperian}}\cdot\mathbf{B}_p}{4\pi}\,d\mathrm{Vol} &= \int \frac{\mathbf{B}_{\mathbf{m},\mathrm{Gilbertian}}\cdot\mathbf{B}_p}{4\pi}\,d\mathrm{Vol} + \int \mathbf{m}\,\delta^3(\mathbf{r})\cdot\mathbf{B}_p\,d\mathrm{Vol} \\ &= -\mathbf{m}\cdot\mathbf{B}_p + \mathbf{m}\cdot\mathbf{B}_p = 0. \end{aligned} \tag{12}$$

That is, keeping track of the delta function at the center of a "point" dipole does not resolve the paradox, as claimed in [12], but reinforces it.

2.2 Interaction Energy of a Dirac String

Paradoxes similar to the above involving magnetic charges were discussed by Lipkin and Peshkin [9, 10], who suggested that they should be considered in the context of Dirac's theory of magnetic charges as being at the end of "strings" of magnetic flux [16, 17].[6] However, Lipkin and Peshkin did not provide a clear resolution of these paradoxes.

We now transcribe an argument by Getino, Rojo and Rubio [13] that the interaction energy between an Ampèrian magnetic dipole and the Dirac string associated with a Gilbertian magnetic charge equals $-\mathbf{m}_\mathrm{A}\cdot\mathbf{B}_p$,[7] which resolves Comay's paradox to the extent that such strings are physical. However, the field energy associated with a pair of Gilbertian magnetic charges then becomes doubtful.[8]

[6]A classical presentation of Dirac strings is given in sec. 6.12 of [2].

[7]Such interaction energy each goes against Dirac's view [17], "supposing each pole to be at the end of an unobservable string".

The argument also associates an infinite self energy with the string, which is distributed along the string. That is, the self energy of a Dirac string is not localized to the magnetic monopole, and a procedure for renormalizing away this infinite energy is unclear. Further, in a closed Universe, the string would essentially fill the entire Universe, with possibly interesting implications. See also Appendix C below.

[8]See [14] for comments by Comay on the argument of Getino *et al.*

2.2.1 Gilbertian Magnetic Charge + Ampèrian Magnetic Moment

Since quantum theories of particles are based on Hamiltonians in which the canonical momentum of a particle with electrical charge q is $\mathbf{p}_{\rm canonical} = \mathbf{p}_{\rm mech} + q\mathbf{A}/c$, where $\mathbf{A}$ is the vector potential of the electromagnetic field acting on the charge, it is desirable that the magnetic field of magnetic charge p be expressible in terms of a vector potential. However, since the magnetic field $\mathbf{B}_p = p\,(\mathbf{r}-\mathbf{r}_p)/\left|\mathbf{r}-\mathbf{r}_p\right|^3$ of a magnetic charge obeys $\boldsymbol{\nabla}\cdot\mathbf{B}_p = 4\pi p\,\delta^3(\mathbf{r}-\mathbf{r}_p)$, we cannot write $\mathbf{B}_p = \boldsymbol{\nabla}\times\mathbf{A}_p$.

Dirac's suggestion [16] was that a magnetic charge p is at the end of an infinite string, here labeled s, and the interior of this string carries magnetic flux $4\pi p$ with sign opposite to that of the flux associated with the "ordinary" magnetic field $\mathbf{B}_p$. That is, denoting $\hat{\mathbf{s}}$ as the unit vector tangent to the string, pointing to the end where the magnetic charge resides, the magnetic field of the string can be written as,

$$\mathbf{B}_s = 4\pi p\,\boldsymbol{\delta}_s, \tag{13}$$

where the vector delta function $\boldsymbol{\delta}_s$ is parallel to $\hat{\mathbf{s}}$, and obeys $\boldsymbol{\delta}_s = 0$ for points not on the string. Furthermore,

$$\int \boldsymbol{\delta}_s\cdot\hat{\mathbf{n}}\,d\text{Area} = \text{sign}(\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}), \tag{14}$$

for an integral over a surface pierced by the string at a point where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

Then, the total magnetic flux across a surface surrounding the magnetic charge is zero, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ where $\mathbf{B} = \mathbf{B}_p + \mathbf{B}_s$, and we can now associate a vector potential $\mathbf{A}_p$ with the field $\mathbf{B}$ of a magnetic charge.

For example, the vector potential of a magnetic charge p at the origin, with Dirac string along the negative-z axis, is,

$$\mathbf{A}_p = p\frac{1-\cos\theta}{r\sin\theta}\,\hat{\boldsymbol{\phi}}, \tag{15}$$

in spherical coordinates (r,θ,ϕ).

Turning to the issue of the interaction energy between a Gilbertian magnetic charge p, taken to be at the origin, and an Ampèrian magnetic dipole $\mathbf{m}_{\rm A}$ at $\mathbf{r}_{\mathbf{m}}$, we write the total field of the magnetic charge as $\mathbf{B}_p + \mathbf{B}_s$, such that,

$$\begin{aligned} U_{\rm int} &= \int \frac{\mathbf{B}_{\mathbf{m}_{\rm A}}\cdot(\mathbf{B}_p+\mathbf{B}_s)}{4\pi}\,d\text{Vol} = \int \frac{\mathbf{B}_{\mathbf{m}_{\rm A}}\cdot\mathbf{B}_s}{4\pi}\,d\text{Vol} = p\int \mathbf{B}_{\mathbf{m}_{\rm A}}\cdot\boldsymbol{\delta}_s\,d\text{Vol} \\ &= p\int_s\int_{\text{surfaces }\perp\text{ to }\hat{\mathbf{s}}} (\mathbf{B}_{\mathbf{m}_{\rm A}}\cdot\boldsymbol{\delta}_s)(d\mathbf{s}\cdot d\mathbf{Area}) \\ &= p\int_s\int_{\text{surfaces }\perp\text{ to }\hat{\mathbf{s}}} (\mathbf{B}_{\mathbf{m}_{\rm A}}\cdot d\mathbf{s})(\boldsymbol{\delta}_s\cdot d\mathbf{Area}) = p\int_s \mathbf{B}_{\mathbf{m}_{\rm A}}\cdot d\mathbf{s} \\ &= -p\int_s \boldsymbol{\nabla}\varphi_{\mathbf{m}_{\rm A}}\cdot d\mathbf{s} = p\,\varphi_{\mathbf{m}_{\rm A}}(0) = p\frac{\mathbf{m}_{\rm A}\cdot -\mathbf{r}_{\mathbf{m}}}{r_{\mathbf{m}}^3} = -\mathbf{m}_{\rm A}\cdot\mathbf{B}_p, \end{aligned} \tag{16}$$

where we have expressed the volume integral of $\mathbf{B}_{\mathbf{m}_{\rm A}}\cdot\boldsymbol{\delta}_s$ as an integral along the Dirac string times integrals over surfaces penetrated by the string; then since $\boldsymbol{\delta}_s$ and $d\mathbf{s}$ are parallel, they can be exchanged in the integrands of the second and third lines; and we note that the magnetic field of the Ampèrian magnetic dipole can be expressed in terms of a magnetic scalar potential $\varphi_{\mathbf{m}_{\rm A}} = \mathbf{m}\cdot(\mathbf{r}-\mathbf{r_m})/\left|\mathbf{r}-\mathbf{r_m}\right|^3$ for points outside the dipole current, with $\mathbf{B}_{\mathbf{m}_{\rm A}} = -\boldsymbol{\nabla}\varphi_{\mathbf{m}_{\rm A}}$. Hence, the interaction energy of the Ampèrian magnetic dipole with the Dirac string associated with the magnetic charge p has the desired value $-\mathbf{m}_{\rm A}\cdot\mathbf{B}_p$ needed to restore conservation of energy, so long as the Dirac string does not pass through the current loop of the Ampèrian dipole.[9]

2.2.2 Two Gilbertian Magnetic Charges

The magnetic field energy of two Gilbertian magnetic chagres, each with an associated Dirac string, would be,

$$\begin{aligned} U_{\rm int} &= \int \frac{(\mathbf{B}_{p_1}+\mathbf{B}_{s_1})\cdot(\mathbf{B}_{p_2}+\mathbf{B}_{s_2})}{4\pi}\, d{\rm Vol} \\ &= \int \frac{\mathbf{B}_{p_1}\cdot\mathbf{B}_{p_2}}{4\pi}\, d{\rm Vol} + \int \frac{\mathbf{B}_{p_1}\cdot\mathbf{B}_{s_2}}{4\pi}\, d{\rm Vol} + \int \frac{\mathbf{B}_{p_2}\cdot\mathbf{B}_{s_1}}{4\pi}\, d{\rm Vol} + \int \frac{\mathbf{B}_{s_1}\cdot\mathbf{B}_{s_2}}{4\pi}\, d{\rm Vol} \\ &= \frac{p_1 p_2}{r_{12}} + p_2\,\varphi_{p_1}(\mathbf{r}_{p_2}) + p_1\,\varphi_{p_2}(\mathbf{r}_{p_1}) = 3\frac{p_1 p_2}{r_{12}}\,, \end{aligned} \qquad (17)$$

assuming that the two Dirac strings do not intersect, so $\mathbf{B}_{s_1}\cdot\mathbf{B}_{s_2} = 0$, and noting that in eq. (16) the field $\mathbf{B}_{\mathbf{m}_{\rm A}}$ could be replaced by the field $\mathbf{B}_p$ of a Gilbertian magnetic charge whose magnetic scalar potential is $\varphi_p(\mathbf{r}) = p/\left|\mathbf{r}-\mathbf{r}_p\right|$ (outside the Dirac string).

However, since the force between two Gilbertian magnetic charges is $\mathbf{F} = p_1 p_2\,\mathbf{r}_{12}/r_{12}^3$, we expect their interaction energy to be $U_{\rm int} = p_1 p_2/r_{12}$.

Hence, the introduction of Dirac strings does not appear to resolve Comay's paradox in the larger sense of accounting for field energy in systems involving Gilbertian magnetic charges (as well as electric currents).

It continues to seem that an electromagnetic field theory in which the field energy is $\int(E^2 + B^2)\, d{\rm Vol}/8\pi$ in not compatible with the existence of both electric and magnetic charges, and that Comay's paradox remains unresolved, at least in a classical context.

2.3 Quantum Analyses

Comay's paradox suggests the despite their appeal, magnetic charges are not compatible with classical electrodynamics.

Present enthusiasm of magnetic charges is in the quantum context, starting with the landmark papers of Dirac [16, 17], and extended to gauge theories by 'tHooft [18] and Polyakov [19].[10] As remarked in sec. 3.1.7 of [23], *There is no classical Hamiltonian theory of magnetic charge.*

[9]Lipkin and Peshkin [9, 10] noted that in case the magnetic charge moves through the current loop, possibly on a trajectory that passes through the loop several times, the Dirac string must become wound around the current loop.

[10]Yang [20]-[22] has emphasized the merits of the language of fiber bundles when discussing monopoles.

Appendix A: Current Loop Maintained by a "Battery"

If the current of an Ampèrian dipole is maintained by a constant-current source ("battery" of appropriately variable voltage V), the latter can contribute to the energy stored in the system, which is therefore not necessarily equal to the work done by the electromagnetic forces.

Current Loop Brought in from Infinity

For example, suppose the system of Gilbertian magnetic charge and Ampèrian magnetic dipole, both at rest, is created by first bringing the magnetic charge in from "infinity" to its final position, and then bringing the magnetic dipole in from "infinity". The field of the magnetic charge does work,

$$W_p = \int_\infty^{\mathbf{r}_m} \mathbf{F}_{\mathbf{m}_\mathrm{A}} \cdot d\mathbf{x}_\mathrm{m} = \int_\infty^{\mathbf{r}_m} \boldsymbol{\nabla}_\mathbf{m}(\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p) \cdot d\mathbf{x}_\mathrm{m} = -\mathbf{m}_\mathrm{A} \cdot \mathbf{B}_p, \tag{18}$$

recalling eq. (7) and noting that the displacement $d\mathbf{x}_\mathrm{m}$ is opposite to the force $\mathbf{F}_{\mathbf{m}_\mathrm{A}}$ on the magnetic dipole as it moves in from "infinity". In addition, the battery does work to maintain constant current I in the Ampèrian loop of (constant) **Area**,

$$W_\mathrm{battery} = \int V_\mathrm{battery} I\, dt = I \int dt\, \frac{d}{dt}\frac{\varphi}{c} = \frac{I\Delta\varphi}{c} = \mathbf{B}_p \cdot \frac{I\,\mathbf{Area}}{c} = \mathbf{B}_p \cdot \mathbf{m}_\mathrm{A}, \tag{19}$$

noting that to keep the current constant the battery must provide a voltage equal and opposite to the $\mathcal{EMF}$ induced in the current loop due to the changing magnetic flux $\varphi = \int \mathbf{B}_p \cdot d\mathbf{Area}$ according to Faraday's law, $\mathcal{EMF}_\mathrm{ind} = -(d/dt)\varphi/c$.[11]

Thus, zero total work is required to assemble the Gilbertian magnetic charge and the Ampèrian magnetic dipole in the above scenario, so it is agreeable that the magnetic field interaction energy (7) is zero in this case.[12]

Current Raised from Zero

Another scenario for assembly of the Gilbertian magnetic charge and Ampèrian magnetic dipole is that initially the current is zero. Then, the charge and loop are brought to their final positions, and the current in loop is raised until its magnetic moment is the desired $\mathbf{m}_\mathrm{A}$. Since the Lorentz force on the current is perpendicular to the latter, no work is done by the field of the magnetic charge as the current is raised. The only work done is that by the "battery" to overcome the back $\mathcal{EMF}$ due to the self inductance of the current loop as the current rises. This results in energy stored in the field of the current loop, which is considered as a "self energy", and not part of the possible interaction energy with the magnetic charge.

[11] In case the field of the magnetic charge flips the direction of the moment $\mathbf{m}_\mathrm{A}$ from antiparallel to parallel $\mathbf{B}_p$, the work done by the field is $2m_\mathrm{A}B_p$, and this energy comes from the "battery", as according to eq. (19), $W_\mathrm{battery} = I\Delta\varphi/c = 2IB_p\mathrm{Area}/c = 2B_p m_\mathrm{A}$. The interaction field energy remains zero during this process.

[12] This argument is given in sec. 5.7 of [2] and on p. 986 of [24].

Hence, also in this scenario, zero work is done while assembling the system that contributes to an interaction field energy.

Thus, Comay's paradox does not apply to Ampèrian magnetic moments that are loops of current maintained by "batteries". The paradox exists only if Nature includes include Gilbertian magnetic charges as well as "permanent" Ampèrian magnetic moments, such as those of electrons, protons and neutrons.[13] Such permanent moments are not well explained in "classical" electrodynamics, and are a feature of quantum electrodynamics.

Hence, Comay's paradox is an aspect of the protrusion of quantum physics into the "classical" realm.

Appendix B: Field Momentum and Angular Momentum

In 1904, J.J. Thomson [25, 26] showed that the field momentum of a magnetic charge and electric charge, both at rest, is zero,

$$\mathbf{P}_{\mathrm{EM}} = \int \mathbf{p}_{\mathrm{EM}}\, d\mathrm{Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = 0, \tag{20}$$

supposing that the field of the magnetic charge is given by $\mathbf{B}_p = p\,(\mathbf{r}-\mathbf{r}_p)/\left|\mathbf{r}-\mathbf{r}_p\right|^3$. He also showed that the field angular momentum of this system is,

$$\mathbf{L}_{\mathrm{EM}} = \int \mathbf{r}\times\mathbf{p}_{\mathrm{EM}}\, d\mathrm{Vol} = \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{q\,p}{c}\,\hat{\mathbf{R}}, \tag{21}$$

where unit vector $\hat{\mathbf{R}}$ points from the electric charge to the magnetic charge.[14]

For systems at rest with fields that fall off sufficiently quickly at large distances, and for which the magnetic field can be deduced from a vector potential, the field momentum and angular momentum can be computed in other ways [30], including,

$$\mathbf{P}_{\mathrm{EM}} = \int \frac{\rho\mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol}, \qquad \mathbf{L}_{\mathrm{EM}} = \int \mathbf{r}\times\frac{\rho\mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol}, \tag{22}$$

where ρ is the electric charge density and $\mathbf{A}^{(\mathrm{C})}$ is the vector potential in the Coulomb gauge.

However, the forms (22) appear to be problematic for the vector potential $\mathbf{A}_p$ associated with the Dirac string of a magnetic charge p, as this would imply,

$$\mathbf{P}_{\mathrm{EM}} = \frac{q\mathbf{A}_p^{(\mathrm{C})}(\mathbf{r}_q)}{c}, \qquad \mathbf{L}_{\mathrm{EM}} = \mathbf{r}_q\times\frac{q\mathbf{A}_p^{(\mathrm{C})}(\mathbf{r}_q)}{c}. \tag{23}$$

such that $\mathbf{P}_{\mathrm{EM}}$ would be nonzero in general, while $\mathbf{L}_{\mathrm{EM}}$ would not point along $\hat{\mathbf{R}} = \hat{\mathbf{r}}_q$ when the magnetic charge is at the origin.

[13] If one associates Dirac strings with magnetic charges to resolve Comay's paradox for permanent Ampèrian magnetic dipoles, then the interaction energy (16) between the dipole and the string violates the conservation of energy that holds in the absence of such strings.

[14] The result (21) was anticipated by Darboux in 1878 [27] and by Poincaré in 1896 [28], but without interpretation of the vector $qp\,\hat{\mathbf{R}}/c$ as the field angular momentum [29].

If the field momentum of a system at rest is nonzero, that system must also contain an equal and opposite "hidden" momentum, such that the total momentum of the system is zero.[15] A system of structureless electric and magnetic charges (at rest) cannot have any "hidden" (internal) momentum, so it is agreeable that the field momentum of this system is zero according to eq. (10). However, if we consider that the magnetic charge is associated with a Dirac string, there is some kind of "hidden" structure to the system, which could then have nonzero field momentum (with equal and opposite "hidden" momentum residing in the string).

Appendix C: Pair Creation of Magnetic Monopoles

Presumably, magnetic monopoles with Dirac strings would not exist in the very early Universe, but would be created as monopole-antimonopole pairs at some later time. The associated strings could not be created with infinite initial length. Rather, a more consistent physical picture would be that a single string connects the monopole and antimonopole, which string grows in length as the two particles separate. Then, the length of the string would never become infinite. However, this scenario would make sense only if there were no self energy associated with the string.

If the string were later broken into two pieces, the newly created ends of the substrings would have to terminate in a monopole or antimonopole. The numbers of monopoles and antimonopoles would always have to be equal, in contrast to ordinary matter for which somehow there exists a dramatic asymmetry in numbers of particles and antiparticles.

It seems unlikely that relatively stable bulk matter made of monopoles or antimonopoles could exist, as the equal numbers of monopoles and antimonopoles would make their annihilation with one another highly probable prior to possible separation into bulk matter and bulk antimatter.[16]

References

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[15] For a review of the concept of "hidden" momentum, see [31].

[16] The magnetic monopoles considered by 'tHooft [18] and Polyakov [19] are not associated with Dirac strings, but are so massive that they are unlikely to form bulk matter. However, their rate of production in the early Universe would be extremely high, unless the evolution of the Universe involved some mechanism, now called inflation [32, 33] to suppress their production.

For a review, see [34].

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