Flow of Energy and Momentum in a Coaxial Cable

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(March 31, 2007)

1 Problem

Discuss the flow of energy and of momentum in, as well as the electromagnetic forces on, a coaxial cable that carries a TEM wave. You may assume the cable to be made from nearly perfect conductors (with a linear medium of dielectric constant ϵ and permeability μ between them), so that the charges and currents are confined to thin layers at the surfaces of the conductors.

This problem is an extension of the case where the coaxial cable carries a steady current [1], for which the discussion concerned the relation between electromagnetic field momentum and "hidden" mechanical momentum in the system.

2 Solution

2.1 Dual Roles of the Poynting Vector and the Maxwell Stress Tensor

This problem illustrates the dual roles of the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{1}$$

(in MKSA units) and the Maxwell stress tensor T,

$$\mathsf{T}_{ij} = E_i D_j + B_i H_j - \frac{1}{2}\delta_{ij}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) = \epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \mu H^2), \tag{2}$$

in a linear medium with dielectric constant ϵ and permeability μ such that $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$.

2.1.1 Energy Balance

As introduced by Poynting [2], the vector $\mathbf{S}$ describes the flow of energy across unit surface area in unit time. In more detail, Poynting noted that the electromagnetic fields do work W on the distributions ϱ and $\mathbf{J} = \varrho\mathbf{v}$ of charge and current density at the rate per unit volume of,[1]

$$\frac{dW}{dt} = \mathbf{f}_{EM} \cdot \mathbf{v} = (\varrho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \cdot \mathbf{v} = \varrho\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \varrho\mathbf{v} \cdot \mathbf{E} = \mathbf{J} \cdot \mathbf{E}, \tag{3}$$

where,

$$\mathbf{f}_{EM} = \varrho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{4}$$

[1]Equation (3) contains the insight that magnetic fields do no work individual charges with no intrinsic magnetic moment.

is the volume density of the fields on the charges and currents. The energy transferred to these charges and currents according to eq. (3) comes from the energy stored in the electromagnetic fields, whose energy density $u_{\rm field}$ according to Maxwell is,

$$u_{\rm field} = \frac{\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}}{2} = \frac{\epsilon E^2 + \mu H^2}{2}\,. \tag{5}$$

The field energy with a volume can also be changed by the flow $\mathbf{S}$ of energy across the surface of that volume, so that conservation of energy can be expressed as,

$$\int \frac{\partial u_{\rm field}}{\partial t}\, d{\rm Vol} = -\oint \mathbf{S}\cdot d\mathbf{Area} - \int \mathbf{J}\cdot\mathbf{E}\, d{\rm Vol} = -\int \boldsymbol{\nabla}\cdot\mathbf{S}\, d{\rm Vol} - \int \mathbf{J}\cdot\mathbf{E}\, d{\rm Vol}, \tag{6}$$

which can be expressed equivalently as the continuity equation,

$$\frac{\partial u_{\rm field}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}. \tag{7}$$

Using Maxwell's equations and various vector calculus identities, Poynting showed that the energy flow vector $\mathbf{S}$ is given by eq. (1).[2]

2.1.2 Momentum Balance

Following the spirit of Poynting's argument, Abraham [9] extended Maxwell's analysis in terms of a stress tensor of electromagnetic forces due to static fields to include the case of time-dependent fields. Recall that for the case of static fields, Maxwell expressed the electromagnetic force $\mathbf{F}_{\rm EM}$ on a volume in terms of an integral of the stress tensor T over the surface of that volume,

$$\mathbf{F}_{\rm EM} = \int \mathbf{f}_{\rm EM}\, d{\rm Vol} = \oint \mathsf{T}\cdot d{\rm Area}. \tag{8}$$

Using Maxwell's (time-dependent) equations and various vector/tensor calculus identities, Abraham showed that eq. (8) can be generalized to the form,[3]

$$\mathbf{F}_{\rm EM} = \int \mathbf{f}_{\rm EM}\, d{\rm Vol} = \oint \mathsf{T}\cdot d{\rm Area} - \int \frac{\partial(\epsilon\mu\mathbf{S})}{\partial t}\, d{\rm Vol}. \tag{9}$$

If the only forces on the charges and currents in the volume are electromagnetic, then Newton's 2$^{\rm nd}$ law can be written as,

$$\mathbf{F}_{\rm EM} = \int \mathbf{f}_{\rm EM}\, d{\rm Vol} = \int \frac{\partial \mathbf{p}_{\rm mech}}{\partial t}\, d{\rm Vol} = \frac{d\mathbf{P}_{\rm mech}}{dt}\,, \tag{10}$$

where $\mathbf{p}_{\rm mech}$ is the volume density of mechanical momentum, and $\mathbf{P}_{\rm mech}$ is the total mechanical momentum in the volume. Combining eqs. (9) and (10) we have,

$$\int \left(\frac{\partial \mathbf{p}_{\rm mech}}{\partial t} + \frac{\partial(\epsilon\mu\mathbf{S})}{\partial t}\right) d{\rm Vol} = \oint \mathsf{T}\cdot d{\rm Area} = \int \boldsymbol{\nabla}\cdot\mathsf{T}\, d{\rm Vol}. \tag{11}$$

[2]Poynting's derivation is discussed in most textbooks on electromagnetism. See, for example, sec. 8.1 of [3], sec. 2.19 of [4], sec. 10-5 of [5], sec. 3.1 and chap. 7 of [6], sec. 6.7 of [7] and sec. 8.1 of [8].

[3]For discussions of Abraham's derivation see, for example, sec. 12.1 of [3], secs. 2.6 and 2.29 of [4], sec. 10-6 of [5], sec. 3.2 and chap. 7 of [6], sec. 6.7 of [7] and sec. 8.2 of [8].

Following Abraham, we identify the vector,

$$\mathbf{p}_{\rm field} = \epsilon\mu \mathbf{S} = \mathbf{D} \times \mathbf{B}, \tag{12}$$

as the momentum per unit volume that is stored in the electromagnetic field, so that the total momentum density is,

$$\mathbf{p}_{\rm total} = \mathbf{p}_{\rm mech} + \mathbf{p}_{\rm field}, \tag{13}$$

which obeys the continuity equation,

$$\frac{\partial \mathbf{p}_{\rm total}}{\partial t} - \boldsymbol{\nabla} \cdot \mathsf{T} = 0. \tag{14}$$

This leads to a second interpretation of the Maxwell stress tensor T, namely that $-\mathsf{T}$ is the flux of momentum in the electromagnetic field. Momentum flux is a tensor, being the vector momentum crossing an (oriented) area element per unit time. Momentum flux has the dimensions of momentum density times velocity (and therefore the same dimensions as energy density and as pressure).

For a TEM plane wave, such as in the present problem, the fields $\mathbf{E}$ and $\mathbf{H}$ and the wave vector $\mathbf{k}$ form an orthogonal triad, the wave velocity is $\mathbf{v} = \hat{\mathbf{k}}/\sqrt{\epsilon\mu}$, the fields obey $\epsilon E^2 = \mu H^2$, the energy density is $u = \epsilon E^2$, the momentum density is $\mathbf{p}_{\rm field} = (u/v)\,\hat{\mathbf{k}}$ and the Maxwell stress tensor has the simple form,

$$\mathsf{T} = -u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -p_{\rm field} v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

if we chose axis 1 along $\mathbf{E}$, axis 2 along $\mathbf{H}$ and axis 3 along $\mathbf{k}$. Interpreting $-\mathsf{T}$ as the momentum flux, we confirm that this flux flows only in the $\mathbf{k}$ direction (across planes perpendicular to $\mathbf{k}$) and has magnitude equal to the momentum density times the wave velocity.

In static or quasistatic examples the Poynting vector, and hence the momentum density, can be zero, while the tensor T is nonzero so long as either the electric or magnetic field is nonzero. In such cases there is formally a momentum flux but no momentum density. Here it is better to consider the tensor T to be simply a measure of the stresses caused by the charges and currents.

In sum, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ has the interpretation as the flux of energy in electromagnetic field, and when multiplied by $\epsilon\mu$ as the density of momentum $\mathbf{p}_{\rm field} = \epsilon\mu\mathbf{S} = \mathbf{D} \times \mathbf{B}$. The Maxwell stress tensor T describes the stresses in a system due to its charges and currents, as well as being the negative of the flux of momentum within the system.[4]

[4]Although we do not need to consider electromagnetic angular momentum here, we note that the vector $\mathbf{r} \times \mathbf{p}_{\rm field}$ describes the density of angular momentum stored in the electromagnetic fields, and the tensor $-\epsilon_{ikl}\, r_k T_{jl}$ describes the flux of angular momentum. See, for example, prob. 5, chap. 3 of [6].

2.2 TEM Wave in a Coaxial Cable with Perfect Conductors

We use a cylindrical coordinate system (r, ϕ, z) with the z axis along the axis of the coaxial cable. The annular gap between the conductor extends from $r = a$ to b, and this gap is filled with a nonconductor with dielectric constant ϵ and permeability μ.

In the idealized case of perfect conductors, the electromagnetic fields are nonzero only for $a < r < b$.

The velocity of the TEM wave is $v = 1/\sqrt{\epsilon\mu}$, and the wave propagates in the $+z$ direction.

2.2.1 E and H Fields for the TEM Wave

The electric field is radial, and can be written as,

$$\mathbf{E} = E_b \frac{b}{r} \cos(kz - \omega t)\,\hat{\mathbf{r}} \qquad (a < r < b), \tag{16}$$

where E_b is the field strength at the outer conductor, $k = 2\pi/\lambda$ and $\omega = kv$. The magnetic field $\mathbf{B}$ has magnitude equal to $E/v = \sqrt{\epsilon\mu}E$, and its direction is azimuthal. The magnetic field $\mathbf{H} = \mathbf{B}/\mu$ can then be written as,

$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}} c\,\hat{\mathbf{z}} \times \mathbf{E} = \sqrt{\frac{\epsilon}{\mu}} E_b \frac{b}{r} \cos(kz - \omega t)\,\hat{\boldsymbol{\phi}} = \frac{E_b}{Z_0}\frac{b}{r} \cos(kz - \omega t)\,\hat{\boldsymbol{\phi}} \qquad (a < r < b), \tag{17}$$

where $Z_0 = \sqrt{\mu/\epsilon} \approx 100\ \Omega$ is the characteristic impedance of the coaxial transmission line.

We recall that the TEM wave fields (16)-(17) consist of the wave function $\cos(kz - \omega t)$ times fields $\mathbf{E}_{\text{static}}$ and $\mathbf{H}_{\text{static}}$ that are possible steady-state fields with no z dependence.

2.2.2 Energy Density

The density (5) of energy stored in the electromagnetic field is,

$$u = \frac{\epsilon E^2 + \mu H^2}{2} = \epsilon E^2 = \epsilon E_b^2 \frac{b^2}{r^2} \cos^2(kz - \omega t). \tag{18}$$

Note that $\epsilon E^2 = \mu H^2$, so the electric and magnetic components of the energy density are equal in the TEM wave.

2.2.3 Poynting Vector

The Poynting vector (1) is,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \sqrt{\frac{\epsilon}{\mu}} E_b^2 \frac{b^2}{r^2} \cos^2(kz - \omega t)\,\hat{\mathbf{z}} = \frac{u}{\sqrt{\epsilon\mu}}\,\hat{\mathbf{z}} = uv\,\hat{\mathbf{z}} \qquad (a < r < b). \tag{19}$$

The Poynting vector equals the electromagnetic energy density times the (vector) wave velocity, which confirms the interpretation of $\mathbf{S}$ as the flux of electromagnetic energy carried by the wave.

2.2.4 Density and Flux of Momentum

The density (12) of momentum stored in the electromagnetic field is,

$$\mathbf{p}_{\rm field} = \epsilon\mu\mathbf{S} = \epsilon\mu\frac{u}{\sqrt{\epsilon\mu}}\,\hat{\mathbf{z}} = \frac{u}{v}\,\hat{\mathbf{z}} \qquad (a < r < b), \tag{20}$$

recalling eq. (19). This momentum density flows along with the wave, so the flux of momentum is the momentum density times the wave velocity,

$$\text{momentum flux} = p_{\rm field}\mathbf{v} = u\,\hat{\mathbf{z}} \qquad (a < r < b). \tag{21}$$

2.2.5 Maxwell Stress Tensor

The Maxwell stress tensor (2) has components in the cylindrical coordinate system,

$$\begin{aligned}
\mathsf{T} &= \begin{pmatrix} T_{rr} & T_{r\phi} & T_{rz} \\ T_{\phi r} & T_{\phi\phi} & T_{\phi z} \\ T_{zr} & T_{z\phi} & T_{zz} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \epsilon E^2 & 0 & 0 \\ 0 & -\epsilon E^2 & 0 \\ 0 & 0 & -\epsilon E^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\mu H^2 & 0 & 0 \\ 0 & \mu H^2 & 0 \\ 0 & 0 & -\mu H^2 \end{pmatrix} \\
&= -\epsilon E^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (a < r < b).
\end{aligned} \tag{22}$$

In the interpretation of $-\mathsf{T}$ as the momentum flux tensor, eq. (22) implies that the only nontrivial component of momentum flux is $-T_{zz} = u = (u/v)v$, which corresponds to momentum density of magnitude u/v flowing with velocity v in the z direction across surfaces perpendicular to the z axis. This is consistent with the previous result (21) for the momentum flux.

2.2.6 Forces on the Coaxial Cable

The stress tensor also has the interpretation as being the electromagnetic force per unit area across an oriented surface. The form of eq. (22) implies that the only nonzero electromagnetic force in the coaxial cable is in the z direction, and this acts across surfaces perpendicular to the z direction.

In greater detail, both the electric and the magnetic parts of the stress tensor (22) have nonzero radial, azimuthal and longitudinal components The radial electric component, $T^E_{rr} = \epsilon E^2/2$, is positive, and implies an attractive force between the opposite charge distributions on the inner and outer conductors of the cable, corresponding to Faraday's insight that there is a tension along the (radial) electric fields lines. The azimuthal and longitudinal electric components, $T^E_{\phi\phi} = T^E_{zz} = -\epsilon E^2/2$, imply that there are repulsive forces between portions of the conductors obtain by, say, slicing them along the planes $x = 0$, $y = 0$ or $z = 0$. These repulsive forces are qualitatively anticipated by Faraday's view that field lines repel one another.

The azimuthal magnetic component $T_{\phi\phi}^{H} = \mu H^2/2$ is positive, and implies that there is an attractive force between filaments of current on the same conductor. The radial and longitudinal magnetic components $T_{rr}^{H} = T_{zz}^{H} = -\mu H^2/2$ are negative and imply that there are repulsive radial magnetic forces between the inner and outer conductor, and also between longitudinal segments of the cable at, say, positive and negative z.

These electric and magnetic forces exist in the static case where the coaxial cable supports a DC voltage or DC current (or both). In the case of a TEM wave, for which $\epsilon E^2 = \mu H^2$, the radial and azimuthal electric and magnetic forces cancel one another, and only the longitudinal forces remain.

This conclusion follows quickly from the form of the Maxwell stress tensor. We now digress to calculate the forces by "elementary" methods.

2.2.7 Cancelation of the Radial Force

The surface density ς of free charge on the inside of the outer conductor can be found from the Maxwell equation $\rho_{\rm free} = \boldsymbol{\nabla} \cdot \mathbf{D} = \epsilon \boldsymbol{\nabla} \cdot \mathbf{E}$, which implies that,

$$\varsigma = \epsilon E_b. \tag{23}$$

The force per unit area on this charge distribution is radially inwards, with magnitude,

$$F_r^E = \frac{\varsigma E_b}{2} = \frac{\epsilon E_b^2}{2}, \tag{24}$$

following the usual argument that the electric field falls to zero from E_b over the small but finite radial thickness of the surface charge distribution, so that the average field on this charge distribution is $E_b/2$.

The azimuthal magnetic field $B_b = \sqrt{\epsilon\mu} E_b$ at the outer conductor acts on the current I in that conductor to produce an outward radial force. From Ampére's law we have that $I = 2\pi b H_b = 2\pi b B_b/\mu$. The force on a portion of the outer conductor of azimthal extent ϕ and length L is $(\phi L/2\pi) I B_b/2$, noting that the magnetic field on the current varies from B_b to zero with average strength $B_b/2$. The area of the portion of the conductor is $\phi b L$, so the outward magnetic force per unit area is,

$$F_r^B = \frac{\phi L}{2\pi} \frac{2\pi b B_b}{\mu} \frac{B_b}{2\phi b L} = \frac{B_b^2}{2\mu} = \frac{\mu H_b^2}{2} = \frac{\epsilon E_b^2}{2} = F_r^E. \tag{25}$$

Thus, the repulsive magnetic force cancels the attractive electric force in the radial direction.

2.2.8 Cancelation of the Transverse Forces

The azimuthal component $T_{\phi\phi}$ of the stress tensor can be used to calculate the transverse force per unit length along the z axis between the portions of the coaxial cable on either side of the plane $x = 0$. In particular, the transverse force dF_x per length dz in the x direction on the portion of the cable at $x < 0$ is given by,

$$dF_x = \int T_{xx}\, dA_x = dz \int_a^b T_{xx}\, dy + dz \int_{-b}^{-a} T_{xx}\, dy = 2\, dz \int_a^b T_{\phi\phi}\, dr, \tag{26}$$

since the area element on the plane $x = 0$ is $dA_x = dy\, dz = dr\, dz$, and on the plane $x = 0$ we have that $T_{xx} = T_{\phi\phi}$.

The electric part of the transverse force per unit length is,

$$\frac{dF_x^E}{dz} = 2\int_a^b T_{\phi\phi}^E\, dr = -2\epsilon\int_a^b E^2\, dr = -2\epsilon\int_a^b E_b^2 \frac{b^2}{r^2}\, dr = -2\epsilon E_b^2 \frac{b(b-a)}{a}\,. \tag{27}$$

The negative sign means that the force is in the $-x$ direction, as expected due to the repulsion between the like charges on the two halves of the conductors.

Similarly, the magnetic part of the transverse force is,

$$\begin{aligned}\frac{dF_x^H}{dz} &= 2\int_a^b T_{\phi\phi}^H\, dr = 2\mu\int_a^b H^2\, dr = 2\mu\int_a^b H_b^2 \frac{b^2}{r^2}\, dr = 2\mu H_b^2 \frac{b(b-a)}{a} \\ &= 2\epsilon E_b^2 \frac{b(b-a)}{a} = -\frac{dF_x^E}{dz}\,.\end{aligned} \tag{28}$$

The total transverse force vanishes because $T_{\phi\phi}^H = -T_{\phi\phi}^E$.

We now attempt to verify the transverse forces (27) and (28) by elementary methods. First, note that the (transverse) electric force per unit length $d\mathbf{F}^E/dz$ on a wire of linear charge density λ that is parallel to the z axis and passes through point $\mathbf{r} = (r, \phi, 0)$ due to a parallel wire of charge density λ' that passes through point $\mathbf{r}' = (r', \phi', 0)$ is,

$$\frac{d\mathbf{F}^E}{dz} = \frac{\lambda\lambda'}{2\pi\epsilon}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\,. \tag{29}$$

We subdivide the inner and outer conductors into wires of azimuthal extent $d\phi$, such that their linear charge densities are $d\lambda_0 = b\epsilon E_b\, d\phi$ on the wire segments of the outer conductor, and $d\lambda_I = -a\epsilon E_a\, d\phi = -b\epsilon E_b b\, d\phi = -d\lambda_O$ on the inner conductor. The portion of the cable at $x < 0$ corresponds to $\pi/2 < \phi < 3\pi/2$, and that at $x > 0$ to $-\pi/2 < \phi < \pi/2$. The x-component of the electric force on the portion of the cable at $x < 0$ is then,

$$\begin{aligned}F_x^E &= \int_{x<0} d\lambda_O \int_{x>0} d\lambda_O' \frac{\cos\phi - \cos\phi'}{4\pi\epsilon b[1-\cos(\phi-\phi')]} + \int_{x<0} d\lambda_I \int_{x>0} d\lambda_I' \frac{\cos\phi - \cos\phi'}{4\pi\epsilon a[1-\cos(\phi-\phi')]} \\ &\quad + \int_{x<0} d\lambda_O \int_{x>0} d\lambda_I' \frac{b\cos\phi - a\cos\phi'}{2\pi\epsilon[a^2+b^2-2ab\cos(\phi-\phi')]} \\ &\quad + \int_{x<0} d\lambda_I \int_{x>0} d\lambda_O' \frac{a\cos\phi - b\cos\phi'}{2\pi\epsilon[a^2+b^2-2ab\cos(\phi-\phi')]} \\ &= (b+a)\frac{b}{a}\frac{\epsilon E_b^2}{4\pi}\int_{\pi/2}^{3\pi/2} d\phi \int_{-\pi/2}^{\pi/2} d\phi' \frac{\cos\phi - \cos\phi'}{1-\cos(\phi-\phi')} \\ &\quad -(b+a)b^2\frac{\epsilon E_b^2}{2\pi}\int_{\pi/2}^{3\pi/2} d\phi \int_{-\pi/2}^{\pi/2} d\phi' \frac{\cos\phi - \cos\phi'}{a^2+b^2-2ab\cos(\phi-\phi')}.\end{aligned} \tag{30}$$

The result of eq. (30) could well be the same as eq. (27). Certainly it is simpler to use the Maxwell stress tensor to obtain the force than it is to use "elementary" methods.

The magnetic force per unit length on the portion of the cable at $x < 0$ can in principle be calculated by an application of the Biot-Savart force law. We subdivide the currents on the conductors into filaments subtending angle $d\phi$, leading to integrals very similar to those in eq. (30).

2.2.9 Significance of the Longitudinal Force on the Cable

The nonzero value of component $T_{zz} = -(\epsilon E^2 + \mu H^2)/2$ of the Maxwell stress tensor (22) implies that there are longitudinal forces on the coaxial cable. These forces exist in the static limit as well.

The electric part of the longitudinal force between portions of the cable at, say, $z < 0$ and $z > 0$ is readily ascribed to the repulsion between the like charges in these two regions. However, it is harder to identify the source of the longitudinal magnetic force, since the currents flow only longitudinally (in the static limit, and also for TEM waves on a cable made of ideal conductors).

Note that the total longitudinal force on any finite portion of the cable, say $z_1 < z < z_2$, vanishes in the static limit, because the longitudinal force on the two ends of this portion is equal and opposite. A nonzero total longitudinal force is obtained for the interval $z_1 < z < z_2$ if one end of the cable lies within this interval, so that $T_{zz} = 0$ at either z_1 or z_2. In this case, there must be radial currents at the termination of the cable, and these radial currents interact with the azimuthal magnetic field to produce the postulated longitudinal force.

DC Current

For example, consider a coaxial cable that extends only for $z < z_0$ and which carries DC current I in the $+z$ direction on its inner conductor and current $-I$ on its outer conductor. The termination at $z = z_0$ is via a uniform resistive disk of thickness d (that extends from $z = z_0$ to $z_0 + d$), so that the radial current density $\mathbf{J}$ in the terminating resistor for $a < r < b$ is,

$$\mathbf{J} = \frac{I}{2\pi r d}\,\hat{\mathbf{r}}. \tag{31}$$

The azimuthal magnetic field at $z = z_0$ is,

$$\mathbf{B} = \frac{\mu I}{2\pi r}\,\hat{\boldsymbol{\phi}}, = \mu H_b \frac{b}{r}\,\hat{\boldsymbol{\phi}}, \tag{32}$$

and falls to zero at $z = z_0 d$. That is, the average magnetic field is 1/2 that of eq. (32),

$$\langle \mathbf{B}(a < r < b, z_0 < z < z_0 + d)\rangle = \mu H_b \frac{b}{2r}\,\hat{\boldsymbol{\phi}}. \tag{33}$$

The magnetic force on the terminating resistor is,

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{B}\,d\mathrm{Vol} = \int_a^b dr \int_0^{2\pi} r\,d\phi \int_{z_0}^{z_0+d} dz\,\frac{H_b b}{rd}\mu H_b \frac{b}{2r}\,\hat{\mathbf{z}} = \mu H_b^2\,\pi b^2 \ln\frac{b}{a}\,\hat{\mathbf{z}}. \tag{34}$$

We compare this with a calculation using the Maxwell stress tensor for a cylinder of radius $R > b$ and longitudinal extent $z_1 < z < z_2$, where $z_1 < z_0$ and $z_2 > z_0 + d$ so that the terminating resistor lies within this interval. Then, the component T_{zz} is $-\mu H^2/2 = -\mu H_b^2 b^2/2r^2$ at z_1 but vanishes at z_2. All components of the tensor T vanish on the cylindrical surface at $r = R$ since this is outside the cable. The (outward pointing) surface area element at $z = z_1$ is $d\mathrm{Area}_z(z_1) = -2\pi r\,dr$, so the total force on the cable within this interval is,

$$\mathbf{F} = \oint \mathsf{T}\cdot d\mathrm{Area} = \int T_{zz}(z_1)\,d\mathrm{Area}_z(z_1)\,\hat{\mathbf{z}} = \int_a^b \left(-\frac{\mu H_b^2 b^2}{2r^2}\right)(-2\pi r\,dr)\,\hat{\mathbf{z}} = \mu H_b^2\,\pi b^2 \ln\frac{b}{a}\,\hat{\mathbf{z}}, \tag{35}$$

as found in eq. (34).

TEM Wave

We return to the case of a TEM wave on the coaxial cable, and calculate the electromagnetic forces on a portion of the cable that does not include the terminating resistor. For this, we apply eq. (9) to a cylinder of radius $R > b$ and longitudinal extent $0 < z < z_1$, for which the relevant (outward pointing) area elements are $d\text{Area}_z(0) = -2\pi r\, dr$ and $d\text{Area}_z(z_1) = 2\pi r\, dr$. Then, recalling eqs. (20)-(22), the force on this portion of the cable is,

$$
\begin{aligned}
\mathbf{F} &= \oint \mathsf{T} \cdot d\text{Area} - \frac{d}{dt}\int \mathbf{p}_{\text{field}}\, d\text{Vol} \\
&= \int T_{zz}(0)\, d\text{Area}_z(0)\, \hat{\mathbf{z}} + \int T_{zz}(z_1)\, d\text{Area}_z(z_1)\, \hat{\mathbf{z}} - \int_a^b 2\pi r\, dr \int_0^{z_1} \frac{\partial(\epsilon\mu\mathbf{S})}{\partial t}\, dz \\
&= \int_a^b \frac{\epsilon E_b^2 b^2}{r^2}[\cos^2\omega t - \cos^2(kz_1 - \omega t)]\, 2\pi r\, dr\, \hat{\mathbf{z}} \\
&\quad - \int_a^b \epsilon\mu\sqrt{\frac{\epsilon}{\mu}} E_b^2 \frac{b^2}{r^2}\, 2\pi r\, dr \int_0^{z_1} 2\omega\cos(kz - \omega t)\sin(kz - \omega t)\, dz\, \hat{\mathbf{z}} \\
&= 2\epsilon E_b^2\, \pi b^2 \ln\frac{b}{a}[\cos^2\omega t - \cos^2(kz_1 - \omega t)]\, \hat{\mathbf{z}} \\
&\quad + 2\epsilon\sqrt{\epsilon\mu}\frac{\omega}{k} E_b^2\, \pi b^2 \ln\frac{b}{a}[\cos^2(kz_1 - \omega t) - \cos^2\omega t]\, \hat{\mathbf{z}} \\
&= 0,
\end{aligned}
\tag{36}
$$

noting that $\omega/k = v = 1/\sqrt{\epsilon\mu}$. Thus, although the forces associated with the Maxwell stress tensor on a portion of the coaxial cable are nonzero, they act to change the electromagnetic field momentum in that portion of the coaxial cable, rather than producing a mechanical force on the conductors of the cable.

The force on a portion of the cable that includes a terminating resistor at $z = z_0$ can be obtained from the analysis contained in eq. (36) by replacing z_1 with z_0 and omitting the contribution from the stress tensor at $z = z_1$,

$$
\mathbf{F} = \epsilon E_b^2\, \pi b^2 \ln\frac{b}{a}\cos^2(kz_0 - \omega t)\, \hat{\mathbf{z}} = 2\mu H_b^2\, \pi b^2 \ln\frac{b}{a}\cos^2(kz_0 - \omega t)\, \hat{\mathbf{z}}. \tag{37}
$$

Another view is that since the terminating resistor absorbs the momentum flowing along the cable, it experiences a force equal to the momentum flux into the resistor, namely,

$$
\mathbf{F} = \int_a^b -T_{zz}(z_0)\, 2\pi r\, dr = \epsilon E_b^2\, \pi b^2 \ln\frac{b}{a}\cos^2(kz_0 - \omega t)\, \hat{\mathbf{z}}. \tag{38}
$$

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