Fields in a Coaxial Cable with a DC Current

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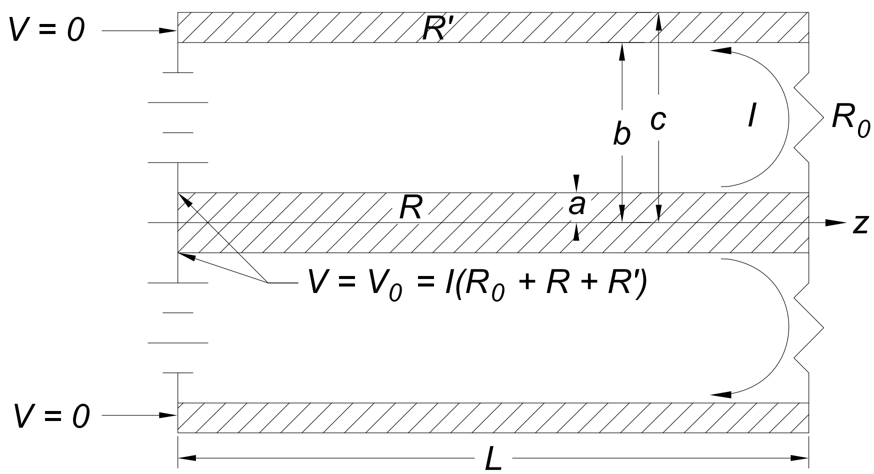
1 Problem

Calculate the electromagnetic fields, and the surface charge densities, of a coaxial cable of length L, whose axis is the z-axis, with inner conductor of radius a, outer conductor that extends from radius b to c, when a battery of voltage difference V_0 is connected to one end and a load resistor R_0 is connected to the other (at larger z). The current may be taken as flowing only in the $+z$ direction inside the conductors and uniformly distributed over them. The conductors have resistivity ϱ, and the battery has negligible internal resistance. Both the battery and the load resistor have the form of annuli.



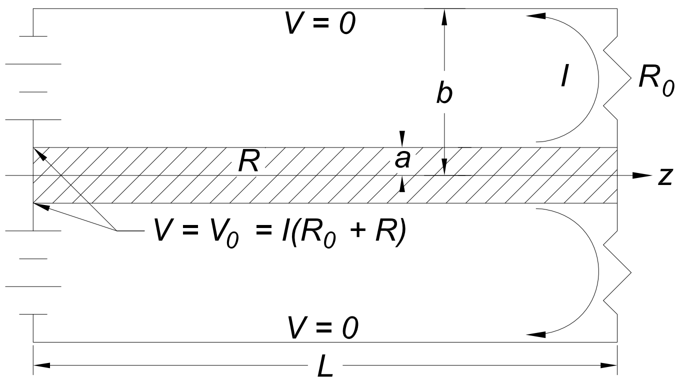
This problem is based on sec. 17 of [1], sec. 10.4b of [2], prob. 7.57, ex. 8.3 and ex. 12.12 of [3].[1]

[1]The fields and Poynting vector found in sec. 2.1 below were discussed qualitatively by Heaviside on p. 212 of [4]; and on pp. 254-55 of the textbook [6], and quantitatively in [7]. See also [8].

2 Solution

2.1 The Outer Conductor Has Zero Resistivity

It is simpler to consider first the case that the outer conductor has zero resistivity, and so is everywhere at electric scalar potential $V = 0$, as in the figure below.[2]



The resistance R of the inner conductor is,

$$R = \frac{\varrho L}{\pi a^2}\,, \tag{1}$$

so the total resistance of the cable plus (annular) load resistor is $R_0 + R$. The DC current I in the system is then,

$$I = \frac{V_0}{R_0 + R}\,. \tag{2}$$

The current, which returns along the outer conductor, causes a magnetic field $\mathbf{B}$ that is nonzero only inside the cable. This field is readily calculated via Ampère's law to be (in Gaussian units, and in a cylindrical coordinate system (r, ϕ, z) with the coaxial cable centered on the z axis),[3]

$$\mathbf{B}(z \text{ inside cable}) = \frac{2I}{c}\hat{\boldsymbol{\phi}} \begin{cases} \frac{r}{a^2} & (r < a), \\ \frac{1}{r} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{3}$$

Inside the inner conductor the electric field is $\mathbf{E}(r < a, z \text{ inside cable}) = IR\,\hat{\mathbf{z}}/L$, as needed to drive the current I against the resistance R.[4] Since the tangential component of the electric field is continuous across a boundary, there must be some electric field in the region $r > a$

[2]As usual in static examples, we work in the Coulomb gauge, which is the same as the Lorenz gauge in such cases. For high frequencies, we advocate use of the Lorenz gauge in "circuit" problems [9].

[3]If $R = 0$ the current flows on the surface of the inner conductor and $\mathbf{B} = 0$ for $r < a$.

[4]If we ignore the resistance R of the inner conductor, an even simpler analysis can be made. The battery can be taken to lie in the plane $z = 0$ and the resistor in the plane $z = L$. For the outer conductor at zero potential, the inner conductor $(r \le a, 0 \le z \le L)$ has $V_0 = IR_0 = V_{\rm battery} = V_{\rm resistor}$, and the electric field is nonzero only inside the cable, $(a < r < b, 0 \le z \le L)$, where it has only the (positive) radial component $E_r = V_0/r\ln(b/a) = -V_0/r\ln(a/b)$. The potential in this region is $V = V_0\ln(r/b)/\ln(a/b)$. The inner conductor has charge $Q = V_0/2\ln(b/a)$ per unit length on its surface.

Much more extensive discussion of this case is given in [10].

as well. Indeed, a charge distribution $Q(z)$ is needed on the surface of the inner conductor to shape the interior electric field to be purely longitudinal.

An analysis of the electric field can be based on the convention that the electric potential $V(r, z)$ is equal to zero on the outer conductor, and is also zero on the plane $z = 0$ (which is not necessarily inside the wire of length L). That is, we suppose the cable extends from $z = -L(1+R_0/R)$ (the position of the battery) to $z = -LR_0/R$ (the position of the resistor), so that the electric potential for $r \le a$ can be written as,

$$V(r \le a, z \text{ inside cable}) = -\frac{IRz}{L}\,. \tag{4}$$

Thus, the potential of the inner conductor at the position of the load resistor is IR_0, and the potential at the connection of the battery to the inner conductor is $I(R_0 + R)$, *i.e.*, the battery voltage (2).

The capacitance C per unit length between the inner and outer conductors of the coaxial cable is well known to be,

$$C = \frac{1}{2\ln(b/a)}\,. \tag{5}$$

The charge $Q(z)$ per unit length on the inner conductor (with charge -$Q(z)$ per unit length on the outer conductor) is therefore,

$$Q(z) = CV(r = a, z) = -\frac{IRz}{2L\ln(b/a)} = \frac{IRz}{2L\ln(a/b)}\,, \tag{6}$$

assuming that $L \gg b$ so that $Q(z)$ is essentially constant over length $\Delta z \ll b$.[5] Further, the potential in the region $a < r < b$ is essentially that for a long wire of charge density $Q(z)$ per unit length, matched to the condition that $V(r = b) = 0$, namely,

$$V(a < r < b, z) = -2Q(z)\ln(r/b) = -\frac{IRz\ln(r/b)}{L\ln(a/b)}\,, \tag{7}$$

which also matches eq. (4) at $r = a$. The potential (7) can also be obtained by a separation-of-variables solution to Laplace's equation [1, 7]; see also sec. 2.2 below.

The electric field $\mathbf{E}$ is obtained by taking the gradient of eq. (7), and we find,

$$\mathbf{E} = \frac{IR}{L}\begin{cases} \hat{\mathbf{z}} & (r < a), \\ \frac{\ln(r/b)}{\ln(a/b)}\hat{\mathbf{z}} + \frac{z}{r\ln(a/b)}\hat{\mathbf{r}} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{8}$$

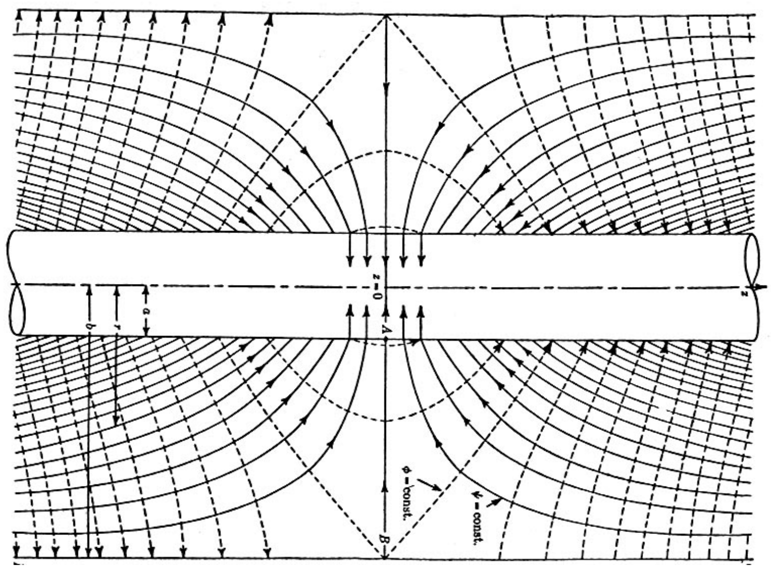
[5] A circuit in the form of a square of edge length L, with battery of potential difference V on one edge and load resistor R_0 on the opposite edge, could be approximated by a coaxial cable of outer radius $b = L$. In this case the charge per unit length (6) implies that a wire segment of length L would have surface charge density $Q/2\pi a \approx -IRz/4\pi aL\ln(L/a) \to -\epsilon_0 IRz/aL\ln(L/a)$, where R is the electrical resistance of that segment, and the latter form holds in SI units. This result was first deduced in 1852 by Weber, secs. 28-36 of item X in [11]. See also sec. 6.2 and Appendix A of [12].

The electromagnetic momentum density $\mathbf{p}_{\rm EM}$ is,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} = \frac{I^2 R}{2\pi c^2 L}\begin{cases} -\frac{r}{a^2}\hat{\mathbf{r}} & (r < a), \\ -\frac{\ln(r/b)}{r\ln(a/b)}\hat{\mathbf{r}} + \frac{z}{r^2\ln(a/b)}\hat{\mathbf{z}} & (a < r < b), \\ 0 & (r > b). \end{cases} \tag{9}$$

The Poynting vector $\mathbf{S}$ quantifies the flow of energy from the battery in the region $(a < r < b, z = -L - R_0/R)$ to the inner conductor and to the load resistor, where the energy is dissipated in Joule heating.

The figure below (from [1]) shows lines of electric field and of Poynting flux in a coaxial cable that has no terminating resistor, but rather is symmetric about the origin and with power sources at both ends. The example considered here corresponds to, say, the left third of the figure, plus a terminating resistive plate; the power source is at the left of the figure.



The total electromagnetic momentum $\mathbf{P}_{\rm EM}$ in the cable is,

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d\text{Vol} = \frac{I^2 R\,\hat{\mathbf{z}}}{2\pi c^2 L\ln(a/b)}\int_a^b 2\pi r\ dr \int_{-L(1+R_0/R)}^{-LR_0/R} dz \frac{z}{r^2} = \frac{I^2 L(R_0 + R/2)}{c^2}\,\hat{\mathbf{z}}. \tag{10}$$

2.2 The Outer Conductor Has Resistivity ϱ

When the outer conductor has resistivity ϱ, its resistance R' is,

$$R' = \frac{\varrho L}{\pi(c^2 - b^2)}\,, \tag{11}$$

and the DC current I in the circuit is,

$$I = \frac{V_0}{R_0 + R + R'}\,. \tag{12}$$

The magnetic field, which is nonzero only inside the cable to a good approximation, is now,

$$\mathbf{B}(r, 0 < z < L) = \frac{2I}{c}\hat{\boldsymbol{\phi}} \begin{cases} \frac{r}{a^2} & (r < a), \\ \frac{1}{r} & (a < r < b), \\ \frac{1}{r}\left(1 - \frac{r^2-b^2}{c^2-b^2}\right) & (b < r < c), \\ 0 & (r > c), \end{cases} \tag{13}$$

where we now suppose that the battery is at $z = 0$ and the load resistor R_0 is at $z = L$.

We also suppose that the flat surfaces of the coaxial cable at $z = 0$ and L with $r < a$ and $b < r < c$ are good conductors, so inside the inner conductor the electric field is again,

$$\mathbf{E}(r < a, 0 < z < L) = \frac{IR\,\hat{\mathbf{z}}}{L}, \tag{14}$$

as needed to drive the current I against the resistance R. Similarly, we expect that the electric field in the outer conductor is $\mathbf{E}(b < r < c, 0 < z < L) = -IR'\,\hat{\mathbf{z}}/L$.

For the electric field $\mathbf{E} = -\boldsymbol{\nabla}V$, we note that the potential V of the center conductor is V_0 at its left end, and $V_0 - IR$ at is right end, while the potential in the outer conductor is zero at its left end, and $IR' = V_0 - I(R + R_0)$ at its right end. The potential inside these conductor should only be a function of z, such that the electric field inside them is longitudinal and uniform, *i.e.*,

$$V(r, 0 < z < L) = \begin{cases} V_0 - \frac{IRz}{L} & (r < a), \\ \frac{IR'z}{L} & (b < r < c). \end{cases} \tag{15}$$

In the region $a < r < b$, $0 < z < L$, taken to be vacuum,[6] the axially symmetric potential $V(r, z)$ obeys Laplace's equation, $\nabla^2 V = 0$, in cylindrical coordinates (r, ϕ, z),

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial z^2} = 0. \tag{16}$$

As usual, we seek solutions that are sums of terms of the separated form $V = F(r)G(z)$, for which eq. (16) implies that,

$$\frac{1}{rF}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \frac{1}{G}\frac{d^2G}{dz^2} = 0, \tag{17}$$

and that,

$$\frac{1}{rF}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = k, \qquad \frac{1}{G}\frac{d^2G}{dz^2} = -k, \tag{18}$$

where k is a separation constant.

[6] If this region were filled with a material of (relative) dielectric constant ϵ, the potential and electric field would be unchanged, but the "free" surface charge density on the cylinders $r = a,\ b$ would be larger than that found in eqs. (34)-(35) by the factor ϵ.

The potential found in sec. 2.1 corresponds to $k = 0$, with $F = A+B \ln r$ and $G = C+Dz$, *i.e.*,

$$V(a < r < b, 0 < z < L) = (A + B \ln r)(C + Dz), \tag{19}$$

with $B = -IR/L \ln(a/b)$, $A = -B \ln b$, $C = 0$ and $D = 1$. Here, we also consider that $k = 0$. Then, the potential (19) at the left end of the inner surface of the outer conductor is,

$$V(b, 0) = 0 = (A + B \ln b)C. \tag{20}$$

We cannot set $C = 0$ as then $V(a, 0)$ would also be zero rather than V. Hence, we must have (if $k = 0$) that,

$$A = -B \ln b. \tag{21}$$

However, this would imply that the potential is zero along the inner surface, $r = b$ of the outer conductor. While this was the case in sec. 2.1, it is not so here.

Instead, we note that a nonzero constant K could be added to the potential (19) with no change in the electric field,

$$V(a < r < b, 0 < z < L) = (A + B \ln r)(C + Dz) + K. \tag{22}$$

Then, the potential at the left end of the inner surface of the outer conductor is,

$$V(b, 0) = 0 = (A + B \ln b)C + K, \qquad AC = -BC \ln b - K. \tag{23}$$

Given that the potential (22) at the left end of the center conductor is V_0, we have that,

$$V(a, 0) = V_0 = (A + B \ln a)C + K. \tag{24}$$

Subtracting eq. (24) from (23), we have that,

$$BC \ln \frac{b}{a} = -V_0. \tag{25}$$

The potential at the right end of the center conductor is $V_0 - IR$, so,

$$V(a, L) = V_0 - IR = (A + B \ln a)(C + DL) + K = V_0 + (A + B \ln a)DL, \tag{26}$$

and,

$$-IR = (A + B \ln a)DL = \frac{(V_0 - K)DL}{C}. \tag{27}$$

Similarly, the potential at the right end of the inner surface of the outer conductor is IR', so,

$$V(b, L) = IR' = (A + B \ln b)(C + DL) + K = (A + B \ln b)DL = -\frac{DKL}{C}. \tag{28}$$

recalling eq. (23). Combining eqs. (27)-(28), we have,

$$-IR = \frac{V_0 DL}{C} + IR', \qquad \frac{DL}{C} = -\frac{R+R'}{R_0+R+R'}\,. \tag{29}$$

From eq. (28),

$$K = -\frac{C}{DL} IR' = \frac{R_0+R+R'}{R+R'} \frac{V_0 R'}{R_0+R+R'} = V_0 \frac{R'}{R+R'}\,, \tag{30}$$

such that K is nonzero if the outer conductor has nonzero resistance R'.

We now have relations for A, B, D and K in terms of C and constant parameters of the problem. Thus, we could, say, take $C = 1$,[7] in which case,

$$A = V_0 \frac{\ln b}{\ln b/a} - V_0 \frac{R'}{R+R'} \qquad B = -\frac{V_0}{\ln b/a}\,, \qquad D = -\frac{1}{L}\frac{R+R'}{R_0+R+R'}\,, \tag{31}$$

and the potential $V(a<r<b, 0<z<L)$ in the gap of the coaxial cable is,

$$V = V_0 \left(\frac{\ln b/r}{\ln b/a} - \frac{R'}{R+R'}\right)\left(1 - \frac{z}{L}\frac{R+R'}{R_0+R+R'}\right) + V_0 \frac{R'}{R+R'}\,. \tag{32}$$

The main point is that a solution exists, which, of course, differs somewhat from that of sec. 2.1 where the resistance of the outer conductor is zero.

The electric field $\mathbf{E}(0<r<c, 0<z<L)$ in the coaxial cable is,

$$\mathbf{E} = \begin{cases} \frac{IR}{L}\,\hat{\mathbf{z}} = \frac{V_0 R}{(R_0+R+R')L}\,\hat{\mathbf{z}} & (r<a), \\ \frac{V_0}{r\ln b/a}\left(1 - \frac{z}{L}\frac{R+R'}{R_0+R+R'}\right)\hat{\mathbf{r}} + V_0\left(\frac{\ln b/r}{\ln b/a} - \frac{R'}{R+R'}\right)\frac{R+R'}{(R_0+R+R')L}\,\hat{\mathbf{z}} & (a<r<b), \\ -\frac{IR'}{L}\,\hat{\mathbf{z}} = -\frac{V_0 R'}{(R_0+R+R')L}\,\hat{\mathbf{z}} & (b<r<c). \end{cases} \tag{33}$$

As expected, E_z is continuous across the surfaces $r = a$ and b.

The potential (15) and (32) is only for the interior of the coaxial cable (plus battery and load resistor). However, the potential is now known everywhere on the surface of this system, so a solution exists (to Laplace's equation) for the exterior region, which could be found by a finite-element analysis if desired. Of course, one also must specify the potential at "infinity", which can reasonably taken to be zero.

Charge densities σ exist on the surfaces $r = a$, b and c for $0<z<L$, as well as on the surfaces $(0<r<c, z=0,\ L)$, the first two of which can be deduced from eq. (33),[8]

$$\sigma(a,z) = 4\pi E_r(a^+,z) = -\frac{4\pi V_0}{a\ln b/a}\left(1 - \frac{z}{L}\frac{R+R'}{R_0+R+R'}\right), \tag{34}$$

$$\sigma(b,z) = -4\pi E_r(b^-,z) = \frac{4\pi V_0}{b\ln b/a}\left(1 - \frac{z}{L}\frac{R+R'}{R_0+R+R'}\right). \tag{35}$$

[7] That is, instead of expressing the potential in terms of four constants A, B C and D as in eq. (19), we use the form (22) with four constants A, B, D and K determined from the known potential at four points, while defining that $C = 1$.

[8] The charge per unit length on the outer surface of the inner conductor is $Q(r = a, z) = 2\pi a\,\sigma(a,z)$, which is equal and opposite to the charge per unit length on the inner surface of the outer conductor, $Q(r = b, z) = 2\pi b\,\sigma(b,z)$.

The other surface charge densities are not determined by the above (interior) solution, but could be computed once the exterior electric potential and field were obtained.[9]

The Poynting vector $\mathbf{S}(0 < r < c, 0 < z < L) = (c/4\pi)\mathbf{E} \times \mathbf{B}$ in the coaxial cable is,

$$\mathbf{S} = \frac{I}{2\pi} \begin{cases} -\frac{IRr}{a^2 L}\,\hat{\mathbf{r}} & (r < a), \\ \frac{V_0}{r^2 \ln b/a} \left(1 - \frac{z}{L}\frac{R+R'}{R_0+R+R'}\right) \hat{\mathbf{z}} - \frac{V_0}{r} \left(\frac{\ln b/r}{\ln b/a} - \frac{R'}{R+R'}\right) \frac{R+R'}{(R_0+R+R')L}\,\hat{\mathbf{r}} & (a < r < b), \\ \frac{IR'}{rL} \left(1 - \frac{r^2-b^2}{c^2-b^2}\right) \hat{\mathbf{r}} & (b < r < c). \end{cases} \tag{36}$$

The power entering resistor R across surface $(r = a, 0 < z < L)$ is $I^2 R$, as expected. Similarly, the power entering resistor R' across surface $(r = b, 0 < z < L)$ is $I^2 R'$, and that entering resistor R_0 across surface $(a < r < b, z = L)$ is,

$$P_{R_0} = \int_a^b 2\pi r\, dr\, S_z(r, L) = \int_a^b 2\pi r\, dr\, \frac{I}{2\pi} \frac{V_0}{r^2 \ln b/a} \frac{R_0}{R_0 + R + R'} = I^2 R_0. \tag{37}$$

Finally, the power delivered by the battery, across surface $(a < r < b, z = 0)$, is,

$$P_{\rm batt} = \int_a^b 2\pi r\, dr\, S_z(r, 0) = \int_a^b 2\pi r\, dr\, \frac{I}{2\pi} \frac{V_0}{r^2 \ln b/a} = I V_0 = I^2 (R_0 + R + R'). \tag{38}$$

2.3 Comment

It was recently claimed in [13] that no consistent solution exists for this problem, which supposedly demonstrated that Ohm's law is not consistent with Maxwell's equations. Whereas, this note illustrates that Ohm and Maxwell are consistent.

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See also pp. 94-95 of [5].

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[9] After all of the (surface) charge densities $\rho(\mathbf{r})$ have been computed, the potential V could also be computed according to $V(\mathbf{r}) = \int d\text{Vol}'\, \rho(\mathbf{r}')/\,|\mathbf{r} - \mathbf{r}'|$, which would confirm the potential found by solving Laplace's equation.

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