Motion of a Pair of Oppositely Charged Spheres

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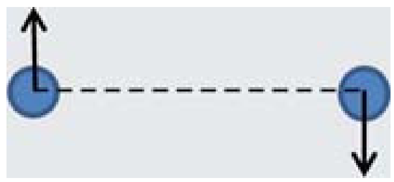
1 Problem

"In outer space, two small balls of equal unknown masses m and charges $\pm q$ are initially held at rest a distance d_0 apart. Then, the balls are simultaneously launched with equal speeds v_0 in opposite directions that are perpendicular to the line connecting the balls. During the subsequent motion of the balls, their minimum speed is $v_{\rm min}$. Find the masses of the balls."



This problem was posed in the Feb. 2012 issue of The Physics Teacher,
http://tpt.aapt.org/resource/1/phteah/v50/i2/p123_s1

2 Solution

This problem is not elementary if one includes relativistic effects, magnetism and radiation by the accelerating charges. The solution given in this section ignores such effects, and also ignores the gravitational interaction between the two masses (although Newtonian gravity could be included with little effect on the following other than increasing the complexity of the form of the potential energy).

The total energy E of the system, approximated as nonrelativistic kinetic energy plus electrostatic potential energy, is constant,

$$E = mv_0^2 - \frac{q^2}{d_0} = mv^2 - \frac{q^2}{d}\,, \tag{1}$$

where each sphere has speed v when their separation is d, and we ignore the energy of the magnetic fields of the moving charges, as well as the energy radiated as a consequence of their acceleration. Gaussian units are employed.

Of course, the center of mass/energy of this system is at rest, at a point we define to be the origin.

Angular momentum is conserved in this problem, but it is complicated to calculate this if we consider the effects of the magnetic fields of the moving charges on one another.

The total energy E can be positive, zero, or negative.

2.1 $E > 0$

When the total energy is positive the spheres eventually move arbitrarily far apart, where their speeds approach $v_{\min}$. The "final" electrostatic energy is zero, so we obtain,

$$mv_0^2 - \frac{q^2}{d_0} = mv_{\min}^2, \tag{2}$$

and hence,

$$m = \frac{q^2}{d_0(v_0^2 - v_{\min}^2)} \qquad (E \geq 0). \tag{3}$$

Angular momentum was not considered in the above analysis, but must be conserved. Hence, the "final" motion of the spheres are along parallel lines, almost radial, offset by distance $d_\perp$ where,

$$v_0 d_0 = v_{\min} d_\perp. \tag{4}$$

The trajectories of the particles are hyperbolae, with one focus at the origin and asymptotically straight lines with separation $d_\perp$.

2.2 $E = 0$

If the total energy is zero the spheres can also move arbitrarily far apart, but now $v_{\min} = 0$, so $m = q^2/d_0 v_0^2$.[1]

The trajectories of the particles in this case are parabolas, with focus at the origin. Asymptotically the trajectories are straight lines parallel to the original line of centers, with infinite offset $d_\perp$ and final velocity $v_{\min} = 0$ such that $d_\perp v_{\min} = d_0 v_0$.

2.3 $E < 0$

If the total energy is negative the spheres cannot move arbitrarily far apart. Rather, their motion is bound, and the trajectories of the spheres are ellipses with a focus at the origin. The initial positions of the spheres are the closest to the origin, where the speed in maximum. The minimum speed occurs when the spheres are the farthest from the origin, and at that point their velocities are perpendicular to the radius vectors from the origin. The maximal separation, $d_{\max}$, of the spheres is related by conservation of angular momentum to the initial conditions,

$$v_0 d_0 = v_{\min} d_{\max}. \tag{5}$$

Using this in the energy equation (1), we have,

$$mv_0^2 - \frac{q^2}{d_0} = mv_{\min}^2 - \frac{q^2 v_{\min}}{v_0 d_0} \tag{6}$$

[1]Formally, eqs. (2)-(4) still apply.

and hence,

$$m = \frac{q^2}{d_0 v_0 (v_0 + v_{\rm min})} \qquad (E < 0). \tag{7}$$

In the limit that $v_{\rm min} \to 0$, $M \to q^2/d_0 v_0^2$ and $d_{\rm max} \to \infty$, as found for the case of $E = 0$.

3 Solution in the Darwin Approximation

To carry the solution to the next approximation, in which terms of order v^2/c^2 are retained, where c is the speed of light in vacuum, we follow Darwin [1]. See also [2]. This approximation includes effects of magnetism, and also of "relativistic mass", but not of radiation.

The Lagrangian for a charge e of (rest) mass m that moves with velocity $\mathbf{v}$ in an external electromagnetic field that is described by potentials ϕ and $\mathbf{A}$ can be written as (see, for example, sec. 16 of [3]),

$$\mathcal{L} = -mc^2 \sqrt{1 - v^2/c^2} - q\phi + q\frac{\mathbf{v}}{c} \cdot \mathbf{A}. \tag{8}$$

Darwin [1] worked in the Coulomb gauge, and kept terms only to order v^2/c^2. Then, the scalar and vector potentials due to a charge q that has velocity $\mathbf{v}$ are (see sec. 65 of [3] or sec. 12.6 of [4]),

$$\phi = \frac{q}{R}, \qquad \mathbf{A} = \frac{q[\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}, \tag{9}$$

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) separation is R.

Combining equations (8) and (9) for a collections of charged particles, and keeping terms only to order v^2/c^2, we arrive at the Darwin Lagrangian,

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{q_i q_j}{R_{ij}} + \sum_{i>j} \frac{q_i q_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right],$$

where we ignore the constant sum of the rest energies of the particles.

The Lagrangian (3) does not depend explicitly on time, so the corresponding Hamiltonian,

$$\mathcal{H} = \sum_i \mathbf{p}_i \cdot \mathbf{v}_i - \mathcal{L}, \tag{10}$$

is the conserved energy of the system, where,

$$\begin{aligned} \mathbf{p}_i &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2}\mathbf{v}_i + \sum_{j \neq i} \frac{q_i q_j}{2c^2 R_{ij}} \left[\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right] \\ &= m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2}\mathbf{v}_i + \sum_{j \neq i} \frac{q_i \mathbf{A}_j(\mathrm{r}_i)}{c} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2}\mathbf{v}_i + \frac{q_i \mathbf{A}_{\mathrm{ext},}(\mathrm{r}_i)}{c} \end{aligned} \tag{11}$$

is the canonical momentum of particle i, and $\mathbf{A}_{\mathrm{ext},i}$ is the vector potential due to charges other than q_i. Hence, the Hamiltonian/energy is,

$$E = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3 m_i v_i^4}{8c^2} + \sum_{i>j} \frac{q_i q_j}{R_{ij}} + \sum_{i>j} \frac{q_i q_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right], \tag{12}$$

as first derived by Darwin [1].

In the present problem there are no external fields, the two particle velocities are equal and opposite at all times, and the (conserved) total energy is,

$$E = mv^2 + \frac{3mv^4}{4c^2} - \frac{q^2}{d} + \frac{q^2}{c^2 d}\left[v^2 + (\mathbf{v}_1 \cdot \hat{\mathbf{n}}_{12})^2\right] = mv_0^2 + \frac{3mv_0^4}{4c^2} - \frac{q^2}{d_0} + \frac{q^2 v_0^2}{c^2 d_0}. \qquad (13)$$

3.1 $E \geq 0$

In this case the spheres travel arbitrarily far apart, such that

$$mv_{\rm min}^2 + \frac{3mv_{\rm min}^4}{4c^2} = mv_0^2 + \frac{3mv_0^4}{4c^2} - \frac{2q^2}{d_0} + \frac{q^2 v_0^2}{c^2 d_0}, \qquad (14)$$

and hence,

$$m = \frac{q^2(1 - v_0^2/c^2)}{d_0(v_0^2 - v_{\rm min}^2)[1 + 3(v_0^2 + v_{\rm min}^2)/4c^2]} \approx \frac{q^2}{d_0(v_0^2 - v_{\rm min}^2)}\left(1 - \frac{7v_0^2 + 3v_{\rm min}^2}{4c^2}\right). \qquad (15)$$

3.2 $E < 0$

The force law for the bounded motion of the two spheres does not vary as $1/r^2$, so the trajectories are not ellipses, but involve a precession that can be large or small depending on the ratio q/m. The minimum velocity does not occur when the masses are along their original line of centers, and is intricate to calculate. We leave this as an exercise for students more energetic than the present author.

There must be some continuity between the solution for $E = 0$ (for which $v_{\rm min} = 0$) and that for $E < 0$ as $E \to 0$ from below. Thus, we anticipate that for $E < 0$,

$$m = \frac{q^2}{d_0 v_0(v_0 + v_{\rm min})}\left(1 - \frac{7v_0^2 + a v_{\rm min}^2}{4c^2}\right) \qquad (E < 0), \qquad (16)$$

for some constant a of order unity.

References

[1] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920),
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