Stress and Momentum in a Capacitor That Moves with Constant Velocity

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1 Problem

Consider a parallel-plate capacitor whose plates are held apart by a nonconducting slab of unit (relative) dielectric constant and unit (relative) magnetic permeability.[1] Discuss the energy, momentum and stress in this (isolated) system when at rest and when moving with constant velocity parallel or perpendicular to the electric field.

Does the system contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined for a subsystem by,

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\, (\mathbf{p} - \rho \mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass," U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[2]

Fringe-field effects can be ignored. The velocity can be large or small compared to the speed of light.

2 Solution

This problem is concerned with the relativistic transformation of properties of the capacitor. It represents a macroscopic application of the concepts of "Poincaré stresses" [3] that were introduced into classical models of the electron. Versions of this problem have also appeared in [4, 5, 6, 7].

We suppose that the capacitor supports a uniform electric field $\mathbf{E}^\star = E^\star\,\hat{\mathbf{z}}$ between its plates, in its rest frame, and we ignore the fringe field.[3] *Taking the (relative) dielectric constant ϵ of the material between the plates to be unity, the electric displacement is given by $\mathbf{D}^\star = \mathbf{E}^\star$ (in Gaussian units). The macroscopic magnetic fields vanish in the capacitor's rest frame, $\mathbf{B}^\star = \mathbf{H}^\star = 0$, noting that the relative permeability is $\mu = 1$.*

[1]The use of unit dielectric constant and unit permeability avoids entering into the interesting controversy as to the so-called Abraham and Minkowski forms of the energy-momentum-stress tensor [1].

[2]The definition (1) was inspired by a private communication from Daniel Vanzella. See also [2].

[3](Mar. 3, 2022) This seems not to be a good approximation for motion with $\mathbf{v} \parallel \mathbf{E}^\star$, as discussed in sec. 2.4 below.

Associated with these electromagnetic fields is the 4-dimensional, macroscopic (symmetric) electromagnetic energy-momentum-stress tensor (secs. 32-33 of [8], sec. 12.10B of [9]),

$$\mathsf{T}^{\mu\nu}_{\rm EM} = \left(\begin{array}{c|c} u_{\rm EM} & c\,\mathbf{p}_{\rm EM} \\ \hline c\,\mathbf{p}_{\rm EM} & -T^{ij}_{\rm EM} \end{array} \right), \tag{2}$$

where indices μ and ν take on values 0, 1, 2, 3, spatial indices i and j take on values 1, 2, 3, $u_{\rm EM}$ is the electromagnetic field energy density,

$$u_{\rm EM} = \frac{E^2 + B^2}{4\pi}, \tag{3}$$

$\mathbf{p}_{\rm EM}$ is the electromagnetic momentum density,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \tag{4}$$

and $T^{ij}_{\rm EM}$ is the 3-dimensional electromagnetic stress tensor.

$$T^{ij}_{\rm EM} = \frac{E_i E_j + B_i B_j}{4\pi} - \delta_{ij} \frac{E^2 + B^2}{8\pi}. \tag{5}$$

In the rest frame of the capacitor the electromagnetic energy-momentum-stress tensor has components,

$$\mathsf{T}^{\star\mu\nu}_{\rm EM} = \left(\begin{array}{c|ccc} \frac{E^{\star 2}}{8\pi} & & \mathbf{0} & \\ \hline & \frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & \frac{E^{\star 2}}{8\pi} & \\ & & & -\frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{6}$$

in the region between the capacitor plates. The nonzero diagonal elements $\mathsf{T}^{\star 11}$, $\mathsf{T}^{\star 22}$ and $\mathsf{T}^{\star 33}$, indicate that there are internal electric forces on, and stresses in, the material between the capacitor plates. Thus, we are led to consider also the mechanical energy-momentum stress tensor,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|c} u_{\rm mech} & c\,\mathbf{p}_{\rm mech} \\ \hline c\,\mathbf{p}_{\rm mech} & -T^{ij}_{\rm mech} \end{array} \right), \tag{7}$$

where the mechanical energy density is $u_{\rm mech} = \rho_m c^2$, the mass/energy density ρ_m includes the term $U_{\rm mech}/c^2$ where $U_{\rm mech}$ is the mechanical energy density associated with nonzero mechanical stresses, the density of mechanical momentum is $\mathbf{p}_{\rm mech}$, and $T^{ij}_{\rm mech}$ is the 3-dimensional mechanical stress tensor.

The capacitor plates are attracted to one another with force/area in the $z^\star$-direction of $E^{\star 2}/8\pi = \mathsf{T}^{\star zz}$. If the material between plates is constrained not to expand transversely, then in the rest frame of the isolated capacitor, the mechanical energy-momentum-stress tensor (in the region between the capacitor plates) has components,[4]

$$\mathsf{T}^{\star\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \rho^\star_m c^2 & & \mathbf{0} & \\ \hline & \frac{\sigma}{1-\sigma}\frac{E^{\star 2}}{8\pi} & & \\ \mathbf{0} & & \frac{\sigma}{1-\sigma}\frac{E^{\star 2}}{8\pi} & \\ & & & \frac{E^{\star 2}}{8\pi} \end{array}\right), \tag{8}$$

where $-1 < \sigma \leq 1/2$ is the so-called Poisson ratio of the medium. $\sigma = 1/2$ corresponds to a perfect fluid/ideal gas, which requires containing walls that also have a mechanical energy-momentum stress tensor. Here, we avoid this complication by supposing that $\sigma = 0$, such that no containing box is required.[5]

The total energy-momentum-stress tensor is the sum of the electromagnetic and mechanical tensors (2) and (7). In the rest frame of the capacitor the total energy-momentum-stress tensor has components,

$$\mathsf{T}^{\star\mu\nu}_{\rm total} = \mathsf{T}^{\star\mu\nu}_{\rm EM} + \mathsf{T}^{\star\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \rho^\star_m c^2 + \frac{E^{\star 2}}{8\pi} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ \mathbf{0} & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ \mathbf{0} & 0 & 0 & 0 \end{array}\right), \tag{9}$$

We interpret the component $\mathsf{T}^{\star 00}$ as implying the total mass density in the rest frame to be,

$$\rho^\star_{\rm total} = \rho^\star_m + \frac{E^2}{8\pi c^2}\,, \tag{10}$$

where $\rho^\star_m$ includes a contribution to the mechanical mass due to the mechanical stress.

2.1 The Capacitor Has Velocity $\mathbf{v} \parallel \mathbf{E}^\star$

In a frame where the capacitor has constant velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the electric and magnetic fields between its plates are given by the transformation,

$$\mathbf{E}_\parallel = \mathbf{E}^\star_\parallel = E^\star\,\hat{\mathbf{z}}, \tag{11}$$

$$\mathbf{E}_\perp = \gamma(\mathbf{E}^\star_\perp - \frac{\mathbf{v}}{c}\times\mathbf{B}^\star) = 0, \tag{12}$$

$$\mathbf{B}_\parallel = \mathbf{B}^\star_\parallel = 0, \tag{13}$$

$$\mathbf{B}_\perp = \gamma(\mathbf{B}^\star_\perp + \frac{\mathbf{v}}{c}\times\mathbf{E}^\star) = 0, \tag{14}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. That is, (ignoring fringe fields) the electromagnetic fields have the same values inside the capacitor in its rest frame and in frames where the capacitor moves

[4]See, for example, eq. (5.13), p. 14 of [10].
[5]The case of an ideal gas in a box is discussed in [11, 12].

with velocity $\mathbf{v}$ parallel to $\mathbf{E}$. Hence, the electromagnetic energy-momentum-stress tensor has the same component values in all such frames,

$$\mathsf{T}^{\mu\nu}_{\rm EM} = \mathsf{T}^{\star\mu\nu}_{\rm EM}. \tag{15}$$

It is useful to confirm this result via a Lorentz transformation of the stress tensor. The transformation L_z from the rest frame to a frame in which the capacitor has velocity $v\,\hat{\mathbf{z}}$ can be expressed in tensor form as,

$$\mathsf{L}_z^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & 0 & 0 & \gamma\beta \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array} \right), \tag{16}$$

where $\beta = v/c$. Then, the transform of a tensor,

$$\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \mathsf{T}^{\star 00} & 0 & 0 & 0 \\ \hline 0 & \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ 0 & 0 & 0 & \mathsf{T}^{\star 33} \end{array} \right), \tag{17}$$

that is diagonal in the rest frame is given by,

$$\mathsf{T}^{\mu\nu} = (\mathsf{L}_z \mathsf{T}^\star \mathsf{L}_z)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathsf{T}^{\star 00} + \gamma^2\beta^2 \mathsf{T}^{\star 33} & 0 & 0 & \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 33}) \\ \hline 0 & \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 33}) & 0 & 0 & \gamma^2\beta^2 \mathsf{T}^{\star 00} + \gamma^2 \mathsf{T}^{\star 33} \end{array} \right). \tag{18}$$

In particular, the transformation of $\mathsf{T}^{\star\mu\nu}_{\rm EM}$, eq. (6), is,

$$\mathsf{T}^{\mu\nu}_{\rm EM} = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1-\beta^2) & 0 & 0 & \gamma^2\beta(1-1) \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma^2\beta(1-1) & 0 & 0 & -\gamma^2(1-\beta^2) \end{array} \right) = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) = \mathsf{T}^{\star\mu\nu}_{\rm EM}, \tag{19}$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathsf{T}^{\star\mu\nu}_{\rm mech}$, eq. (8), is, with $\sigma = 0$,

$$\mathsf{T}^{\mu\nu}_{\rm mech} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_m c^2 + \gamma^2\beta^2 \frac{E^{\star 2}}{8\pi} & 0 & 0 & \gamma^2\beta\left(\rho^\star_m c^2 + \frac{E^{\star 2}}{8\pi}\right) \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma^2\beta\left(\rho^\star_m c^2 + \frac{E^{\star 2}}{8\pi}\right) & 0 & 0 & \gamma^2\beta^2\rho^\star_m c^2 + \gamma^2\frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{20}$$

and the transformation of the total stress tensor $\mathsf{T}^{\star\mu\nu}_{\rm total}$, eq. (9), is,

$$\mathsf{T}^{\mu\nu}_{\rm total} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_{\rm total} c^2 & 0 & 0 & \gamma^2\beta\rho^\star_{\rm total} c^2 \\ \hline 0 & \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ 0 & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ \gamma^2\beta\rho^\star_{\rm total} c^2 & 0 & 0 & \gamma^2\beta^2\rho^\star_{\rm total} c^2 \end{array} \right) = \mathsf{T}^{\mu\nu}_{\rm EM} + \mathsf{T}^{\mu\nu}_{\rm mech}, \tag{21}$$

with the total mass density $\rho^\star_{\rm total}$ given by eq. (10).

One noteworthy feature of eq (21) is the nonzero value of $\mathsf{T}^{33}_{\rm total}$. We recall that the purely spatial components of a stress-energy-momentum tensor have the dual interpretation as the momentum-flux tensor. In the present case, the flux of momentum is in the z direction, with magnitude equal to the momentum density times v, namely $(\gamma\rho^\star_{\rm total} v)\cdot v = \gamma^2\beta^2\rho^\star_{\rm total}c^2 = \mathsf{T}^{33}_{\rm total}$.

Turning to the component $\mathsf{T}^{00}_{\rm total}$, we note that the mass of the material between the capacitor plates, when moving with velocity $\mathbf{v}$, is larger than its rest mass by the factor γ. However, a moving volume element is smaller by the factor $1/\gamma$ than when that element is at rest. Hence, the mass density ρ_m is larger by a factor of γ^2 for the moving capacitor than when at rest,

$$\rho_m = \gamma^2 \rho^\star_m. \tag{22}$$

Thus, the component $\mathsf{T}^{00}_{\rm total}$ transforms as expected for a mass density.

Furthermore, the four components,

$$(\mathsf{T}^{00}_{\rm total}, \mathsf{T}^{01}_{\rm total}, \mathsf{T}^{02}_{\rm total}, \mathsf{T}^{03}_{\rm total}) \tag{23}$$

of the total energy-momentum-stress tensor transform as an energy-momentum-density 4-vector, although this is not the case for the sets of components,

$$(\mathsf{T}^{00}_{\rm EM}, \mathsf{T}^{01}_{\rm EM}, \mathsf{T}^{02}_{\rm EM}, \mathsf{T}^{03}_{\rm EM}) \qquad \text{or} \qquad (\mathsf{T}^{00}_{\rm mech}, \mathsf{T}^{01}_{\rm mech}, \mathsf{T}^{02}_{\rm mech}, \mathsf{T}^{03}_{\rm mech}) \tag{24}$$

separately. This illustrates a general result that within volumes that contain both electromagnetic fields and matter, the concepts of electromagnetic momentum density, $\mathsf{T}^{0i}_{\rm EM}/c = \mathbf{E}\times\mathbf{B}/4\pi c$, and mechanical momentum density, $\mathsf{T}^{0i}_{\rm mech}/c$, are not consistent with being components of a energy-momentum 4-vector; only the total momentum density, $\mathsf{T}^{0i}_{\rm total}/c$, is satisfactory in this respect. The great utility of the concept of "electromagnetic" momentum in matter-free regions leads us to attach similar significance to it in systems containing matter. However, this often results in difficulties in the interpretation of the "mechanical" part of the momentum.

As noted in [5], $\mathsf{T}_{\rm EM}^{\star 0i} = 0$, so there is no flow of electromagnetic energy along with the moving capacitor when $\mathbf{v} \parallel \mathbf{E}$. However, the flow of mechanical energy density associated with the moving capacitor is $c\mathsf{T}_{\rm mech}^{\star 0i}$, and we see in eq.(20) that $c\mathsf{T}_{\rm mech}^{\star 03}$ has a term $\gamma^2 E^{\star 2}\mathbf{v}/8\pi = \mathsf{T}_{\rm EM}^{\star 00}\mathbf{v}$, which is the value perhaps naïvely expected for $c\mathsf{T}_{\rm EM}^{\star 0i}$. That is, the flow of energy in the moving capacitor is "mechanical," not "electromagnetic." The electromagnetic field energy inside the moving capacitor is at rest, while the "bottom" plate of the capacitor "sweeps up" this energy, converts it to mechanical energy that flows up to the "top" plate inside the stressed dielectric, where it is converted back into electromagnetic energy. This is an example of the relativity of steady energy flow [13, 14, 15].

For a capacitor moving with $\mathbf{v} \parallel \mathbf{E}$ the electromagnetic-field momentum is zero,

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d\mathrm{Vol} = 0, \tag{25}$$

in that $\mathsf{T}_{\rm EM}^{0i} = c\,p_{{\rm EM},i} = 0$, while the electromagnetic field energy is $U_{\rm EM} = \mathsf{T}_{\rm EM}^{00} V = E^{\star 2}V^\star/8\pi\gamma = U_{\rm EM}^\star/\gamma$, where $V = V^\star/\gamma$ is the Lorentz-contracted volume of the moving capacitor.[6] The effective mass of this energy is $M_{\rm EM} = E^{\star 2}V/8\pi c^2 = E^{\star 2}V^\star/8\pi\gamma c^2 = M_{\rm EM}^\star/\gamma$, and it moves with velocity $\mathbf{v}$, so that,

$$M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{E^{\star 2}V}{8\pi c^2}\mathbf{v}. \tag{26}$$

Thus, according to the definition (1) the electromagnetic field of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,EM} = -\frac{E^{\star 2}V}{8\pi c^2}\mathbf{v}. \tag{27}$$

This momentum is "hidden" in the sense that the electromagnetic field has no momentum, but its center of mass/energy is in motion.

The mechanical momentum density is given from eq. (20) as,

$$\mathbf{p}_{\rm mech} = \gamma^2\left(\rho_m^\star + \frac{E^{\star 2}}{8\pi c^2}\right)\mathbf{v} = \rho_{\rm total}\mathbf{v}. \tag{28}$$

The mechanical momentum is the momentum density times the volume V,

$$\mathbf{P}_{\rm mech} = \mathbf{p}_{\rm mech}V = \gamma^2 V\left(\rho_m^\star + \frac{E^{\star 2}}{8\pi c^2}\right)\mathbf{v}. \tag{29}$$

The mechanical mass density is,

$$\boldsymbol{\rho}_{\rm mech} = \mathsf{T}_{\rm mech}^{00} = \gamma^2\left(\rho_m^\star + \beta^2\frac{E^{\star 2}}{8\pi c^2}\right), \tag{30}$$

[6](Dec. 8, 2020) Rohrlich [16] advocated an electromagnetic energy momentum 4-vector $P_{{\rm Rohrlich},\mu} = \gamma\, U_{\rm EM}^\star(1, \mathbf{v}/c) = (U_{\rm EM,Rorhlich}, c\,\mathbf{P}_{\rm EM,Rorhlich})$, *i.e.*, $U_{\rm EM,Rorhlich} = \gamma\, U_{\rm EM}^\star$ and $\mathbf{P}_{\rm EM,Rorhlich} = \gamma\, U_{\rm EM}^\star \mathbf{v}/c^2$. This formalism makes little physical sense to the present author.

Note also that $U_{\rm EM,Rorhlich} = \gamma^2\, U_{\rm EM}$; Rohrlich's eqs. (3.25) and (3.26) should have factors of γ^2 not γ.

and this moves with velocity $\mathbf{v}$ such that,

$$M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \boldsymbol{\rho}_{\rm mech}V\mathbf{v} = \gamma^2 V\left(\rho_m^\star + \beta^2\frac{E^{\star 2}}{8\pi c^2}\right). \tag{31}$$

According to definition (1) the matter of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm mech} - M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \frac{E^{\star 2}V}{8\pi c^2}\mathbf{v} = -\mathbf{P}_{\rm hidden,EM}. \tag{32}$$

This result reflects that the energy and momentum of stress in a moving subsystem do not transform like a 4-vector if that subsystem interacts with another subsystem (here, the electromagnetic field).

The total "hidden" momentum of the system is zero.[7]

2.2 The Capacitor Has Velocity $\mathbf{v} \perp \mathbf{E}^\star$

In a frame where the capacitor has constant velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, the electric and magnetic fields between its plates are given by the transformation,

$$\mathbf{E}_\parallel = \mathbf{E}_\parallel^\star = 0, \tag{33}$$

$$\mathbf{E}_\perp = \gamma(\mathbf{E}_\perp^\star - \frac{\mathbf{v}}{c}\times\mathbf{B}^\star) = \gamma E^\star\,\hat{\mathbf{z}}, \tag{34}$$

$$\mathbf{B}_\parallel = \mathbf{B}_\parallel^\star = 0, \tag{35}$$

$$\mathbf{B}_\perp = \gamma(\mathbf{B}_\perp^\star + \frac{\mathbf{v}}{c}\times\mathbf{E}^\star) = -\gamma\frac{v}{c}E^\star\,\hat{\mathbf{y}}. \tag{36}$$

Using eqs. (2)-(5) together with eqs. (33)-(36), the electromagnetic energy-momentum-stress tensor of the moving capacitor is,

$$\mathsf{T}_{\rm EM}^{\mu\nu} = \frac{E^{\star 2}}{8\pi}\left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right). \tag{37}$$

We confirm this result using the Lorentz transformation L_x from the rest frame to a frame in which the capacitor has velocity $v\,\hat{\mathbf{x}}$,

$$\mathsf{L}_x^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & \gamma\beta & 0 & 0 \\ \hline \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{38}$$

[7]If we consider the system to consist of two subsystems, A = capacitor plates and charge thereon, B = dielectric + electromagnetic fields, then subsystem B has the same properties as the entire system considered above (where we neglected the mass/energy of the capacitor plates). Hence, subsystem B has zero "hidden" momentum.

Then, the transform of a tensor (17) that is diagonal in the rest frame is given by,

$$\mathsf{T}^{\mu\nu} = (\mathsf{L}_x \mathsf{T}^\star \mathsf{L}_x)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2 \mathsf{T}^{\star 00} + \gamma^2\beta^2 \mathsf{T}^{\star 11} & \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 11}) & 0 & 0 \\ \hline \gamma^2\beta(\mathsf{T}^{\star 00} + \mathsf{T}^{\star 11}) & \gamma^2\beta^2 \mathsf{T}^{\star 00} + \gamma^2 \mathsf{T}^{\star 11} & 0 & 0 \\ 0 & 0 & \mathsf{T}^{\star 22} & 0 \\ 0 & 0 & 0 & \mathsf{T}^{\star 33} \end{array} \right), \tag{39}$$

In particular, the transformation of $\mathsf{T}^{\star\mu\nu}_{\mathrm{EM}}$, eq. (6), is,

$$\mathsf{T}^{\mu\nu}_{\mathrm{EM}} = \frac{E^{\star 2}}{8\pi} \left(\begin{array}{c|ccc} \gamma^2(1+\beta^2) & 2\gamma^2\beta & 0 & 0 \\ \hline 2\gamma^2\beta & \gamma^2(1+\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right), \tag{40}$$

as found above.

Similarly, the transformation of the mechanical stress tensor $\mathsf{T}^{\star\mu\nu}_{\mathrm{mech}}$, eq. (8), is, with $\sigma = 0$,

$$\mathsf{T}^{\mu\nu}_{\mathrm{mech}} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_m c^2 & \gamma^2\beta\rho^\star_m c^2 & 0 & 0 \\ \hline \gamma^2\beta\rho^\star_m c^2 & \gamma^2\beta^2\rho^\star_m c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{E^{\star 2}}{8\pi} \end{array} \right), \tag{41}$$

the transformation of the total stress tensor $\mathsf{T}^{\star\mu\nu}_{\mathrm{total}}$, eq. (9), is,

$$\mathsf{T}^{\mu\nu}_{\mathrm{total}} = \left(\begin{array}{c|ccc} \gamma^2 \rho^\star_{\mathrm{total}} c^2 + \gamma^2\beta^2 \frac{E^{\star 2}}{8\pi} & \gamma^2\beta\rho^\star_{\mathrm{total}} c^2 + +\gamma^2\beta \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ \hline \gamma^2\beta\rho^\star_{\mathrm{total}} c^2 + \gamma^2\beta \frac{E^{\star 2}}{8\pi} & \gamma^2\beta^2\rho^\star_{\mathrm{total}} c^2 + \gamma^2 \frac{E^{\star 2}}{8\pi} & 0 & 0 \\ 0 & 0 & \frac{E^{\star 2}}{8\pi} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = \mathsf{T}^{\mu\nu}_{\mathrm{EM}} + \mathsf{T}^{\mu\nu}_{\mathrm{mech}}, \tag{42}$$

with the total mass density $\rho^\star_{\mathrm{total}}$ again given by eq. (10), $\rho^\star_{\mathrm{total}} = \rho^\star_m + \frac{E^{\star 2}}{8\pi c^2}$.[8]

[8]The top row of eq. (42) shows that the flow of total energy between the capacitor plates equals the energy density there times the velocity of the capacitor. However, the top rows of eqs. (40)-(41) indicate that this is not true separately for the electromagnetic and mechanical energies. If the dielectric between the capacitor plates does not cover the entire area of the plates, such that there are regions in which the only form of energy density is electromagnetic, then the flow of energy here is not simply the energy density times the bulk velocity of the capacitor. Since this is a steady-state example, there must be a closed circulation of energy flow, which therefore includes some flow opposite to the direction of motion of the capacitor. To account for this circulation, we must consider both the fringe field of the capacitor, and the mechanical energy density inside the (stressed) plates of the capacitor [5]. This more complicated variant has counterintuitive aspects also encountered is examples such as the belt drive considered by Taylor and Wheeler [13, 14]; see also [15].

For a capacitor moving with $\mathbf{v} \perp \mathbf{E}$ the electromagnetic-field momentum is found from from eq. (40) to be,

$$\mathbf{P}_{\rm EM} = \frac{\mathsf{T}^{01}_{\rm EM} V}{c}\,\hat{\mathbf{x}} = \frac{2\gamma^2 E^{\star 2} V}{8\pi c^2}\mathbf{v}, \tag{43}$$

and,

$$M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{\mathsf{T}^{00}_{\rm EM} V}{c^2}\mathbf{v} = \frac{\gamma^2(1+\beta^2) E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{44}$$

According to definition (1) the electromagnetic field of the moving capacitor possesses "hidden" momentum,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{45}$$

The mechanical momentum is given from eq. (41) as,

$$\mathbf{P}_{\rm mech} = \frac{\mathsf{T}^{01}_{\rm mech} V}{c}\,\hat{\mathbf{x}} = \gamma^2 V \rho^\star_m\,\mathbf{v}, \tag{46}$$

and,

$$M_{\rm mech}\mathbf{v}_{\rm cm,mech} = \frac{\mathsf{T}^{00}_{\rm mech} V}{c^2}\mathbf{v} = \gamma^2 V \rho^\star_m\,\mathbf{v}. \tag{47}$$

According to definition (1) the matter of the moving capacitor possesses no "hidden" momentum,

$$\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm mech} - M_{\rm mech}\mathbf{v}_{\rm cm,mech} = 0. \tag{48}$$

The total "hidden" momentum of the system is nonzero,

$$\mathbf{P}_{\rm hidden,total} = \mathbf{P}_{\rm hidden,EM} + \mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm hidden,EM} = \frac{E^{\star 2} V}{8\pi c^2}\mathbf{v}. \tag{49}$$

2.3 Comments

This example is noteworthy in that there is no "hidden" momentum in either the mechanical or electromagnetic subsystems when the system is at rest, while the total "hidden" momentum is nonzero when the system has uniform motion with $\mathbf{v}$ perpendicular to $\mathbf{E}^\star$, but zero total "hidden" momentum for uniform motion with $\mathbf{v}$ parallel to $\mathbf{E}^\star$.

If we had followed [11, 12] in supposing that the medium between the capacitor plates is a gas, then $\mathsf{T}^{\star 11}_{\rm mech} = \mathsf{T}^{\star 22}_{\rm mech} = E^{\star 2}/8\pi$ rather than zero as assumed in this note. This change has no effect on the results of sec. 2.1 above, but in sec. 2.2 it would alter eqs. (46)-(48) such that $\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm hidden,EM}$, and the total "hidden" momentum of the moving system would again be nonzero, but with a value twice that than found in eq. (49). However, this conclusion ignores the stress-energy-momentum in the box that contains the gas.

2.4 Fringe Fields (Mar. 3, 2022)

The preceding analysis neglected the fringe fields of the capacitor.

For some understanding of the consequences of this neglect, we recall that for quasistatic systems, the electromagnetic field momentum can be computed several ways [18],

$$\mathbf{P}_{\rm EM} = \int \frac{\varrho \mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\mathbf{r}\, d{\rm Vol}, \qquad (50)$$

where the potentials $\mathbf{A}^{\rm (C)}$ and $V^{\rm (C)}$ are in the Coulomb gauge.

We suppose the capacitors has square plates of edge length $l^\star$ parallel to the x-y plane, $0 < x^\star < l^*$, $0 < y^\star < l^*$, at $z^* = 0$ and d^* in the rest frame of the capacitor. We approximate the electric charge density on the plates as uniform with surface charge density $\sigma^* = E^\star/4pi$, where $E^\star$ is the magnitude of the electric field in the capacitor if edge effects are ignored.

In the lab frame the capacitor has velocity $\mathbf{v}$, either parallel or perpendicular to the lab-frame field $\mathbf{E}^\star = E^\star\,\hat{\mathbf{z}}$. Then, at time $t = 0$, when the moving capacitor has the positive plate at $z = 0$ and the negative plate at $z = d$,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \int \frac{\varrho \mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} = \int_0^{l_x}\int_0^{l_y} dx\, dy\, \frac{\sigma_+\mathbf{A}^{\rm (C)}(x,y,0) + \sigma_-\mathbf{A}^{\rm (C)}(x,y,d)}{c} \\
&= \int_0^{l_x}\int_0^{l_y}\int_0^{l_x}\int_0^{l_y} dx\, dy\, dx'\, dy'\, \frac{\sigma_+}{c}\left(\frac{\sigma_+\mathbf{v}}{c\sqrt{(x-x')^2+(y-y')^2}} + \frac{\sigma_-\mathbf{v}}{c\sqrt{(x-x')^2+(y-y')^2+d^2}}\right) \\
&+ \int_0^{l_x}\int_0^{l_y}\int_0^{l_x}\int_0^{l_y} dx\, dy\, dx'\, dy'\, \frac{\sigma_-}{c}\left(\frac{\sigma_+\mathbf{v}}{c\sqrt{(x-x')^2+(y-y')^2+d^2}} + \frac{\sigma_-\mathbf{v}}{c\sqrt{(x-x')^2+(y-y')^2}}\right) \\
&= -\frac{2\sigma^2\mathbf{v}}{c^2}\int_0^{l_x}\int_0^{l_y}\int_0^{l_x}\int_0^{l_y} dx\, dy\, dx'\, dy'\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+d^2}}\right) \\
&= \int_0^{l_x}\int_0^{l_y} dx\, dy\, \frac{\sigma_+\mathbf{v}V^{\rm (C)}(x,y,0) + \sigma_-\mathbf{v}V^{\rm (C)}(x,y,d)}{c} = \int \frac{\mathbf{J}V^{\rm (C)}}{c}\, d{\rm Vol}. \qquad (51)
\end{aligned}$$

For low velocities, $l_x \approx l_y \approx l^*$ and $d \approx d^*$, so the above forms are independent of the direction of $\mathbf{v}$ in this regime.[9] This suggests that the analysis in sec. 2.2 above, for $\mathbf{v} \perp \mathbf{E}^\star$, is more correct than that in sec. 2.1, for $\mathbf{v} \parallel \mathbf{E}^\star$.

For a simplified illustration of the above, we consider a "capacitor" that consists of only two charges $\pm q$ held apart at fixed distance $d^\star\,\hat{\mathbf{z}}$ in the rest frame of that system.[10] Then, in a lab frame where the system has velocity $\mathbf{v}$ either parallel or perpendicular to the x-axis, and the charges are separated by distance d, Then, at time $t = 0$, when the moving capacitor has the positive plate at $z = 0$ and the negative plate at $z = d$,

$$\mathbf{P}_{\rm EM} = \frac{q\mathbf{A}_-^{\rm (C)} - q\mathbf{A}_+^{\rm (C)}}{c} = \frac{q}{c}\frac{-q\mathbf{v}}{cd} - \frac{q}{c}\frac{q\mathbf{v}}{cd} = -\frac{2q^2}{c^2 d}\mathbf{v}, \qquad (52)$$

which is also independent of the direction of $\mathbf{v}$ for low velocities.

[9]Thanks to V. Onoochin for pointing this out.

[10]This model is discussed in [19].

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