Energy Balance while Charging a Capacitor

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1 Problem

Discuss the energy balance during the charging of a capacitor by a battery in a series R-C circuit. Comment on the limit of zero resistance.[1]

2 Solution

The loop equation[2] for a series R-C circuit,[3] driven by a battery of voltage drop V, is,

$$V = IR + \frac{Q}{C}\,, \tag{1}$$

where the current I is related to the charge Q on the capacitor plates by $I = dQ/dt \equiv \dot{Q}$. The time derivative of eq. (1) is,

$$0 = \dot{I}R + \frac{I}{C}\,, \tag{2}$$

whose solution is,

$$I(t > 0) = \frac{V}{R}\, e^{-t/RC}, \tag{3}$$

supposing that the current starts to flow at time $t = 0$. The final charge on the capacitor is,

$$Q_{\rm final} = CV. \tag{4}$$

The energy delivered by the battery as the current flows is,

$$\Delta U_{\rm batt} = \int_0^\infty VI\,dt = V\frac{V}{R}RC = CV^2, \tag{5}$$

which is independent of the value of the resistance R. This result can be deduced another way, by noting that the battery has moved charge $Q_{\rm final}$ across potential difference V as the capacitor charged, so it did work,

$$W = Q_{\rm final}V = CV^2 = \Delta U_{\rm batt}. \tag{6}$$

[1]This problem is related to the so-called "two-capacitor paradox" [1].

[2]The loop equation is generally ascribed to Kirchhoff [2, 3], although he only considered circuits with batteries and resistors. The present form of "Kirchhoff's" loop equation for circuits that also include electric generators, capacitors and inductors is due to Maxwell [4], following earlier discussion by W. Thomson [5] of the discharge of a capacitor through a resistor.

[3]Any internal, series resistance of the voltage source is included in the resistance R.

The energy dissipated as heat in the resistor is,

$$U_R = \int_0^\infty I^2 R\, dt = \frac{V^2}{R^2} R \frac{RC}{2} = \frac{CV^2}{2}\,, \tag{7}$$

which also is independent of R. The energy stored in the capacitor at $t \to \infty$ is,

$$U_C = \frac{CV^2}{2}\,. \tag{8}$$

Energy is conserved in this process,

$$\Delta U_{\rm batt} = U_R + U_C. \tag{9}$$

3 The Case of Zero Resistance

If we consider the case of zero resistance R, we have a paradox, in that eq. (6) still holds for the energy delivered by the battery, but now it would seem that $U_R = 0$ and energy is not conserved, since $\Delta U_{\rm batt} = 2U_C$.[4]

The resolution of the paradox is that as R goes to zero, the acceleration of charges in the circuit, which is proportional to $\dot I = V\, e^{-t/RC}/R^2 C$, is very large at small t, and radiation cannot be ignored. Since radiation dissipates energy of the battery, the circuit can be thought of as containing an additional series resistance $R_{\rm rad}$, which, while generally small, is nonzero. Then, the total series resistance in the circuit is,

$$R_{\rm total} = R + R_{\rm rad}, \tag{10}$$

and the resistance R in eqs. (1)-(5) should be replaced by $R_{\rm total}$, such that,

$$U_{R_{\rm total}} = \frac{CV^2}{2}\,, \tag{11}$$

even when $R = 0$. That is, eq. (9) should always have been,

$$\Delta U_{\rm batt} = U_{R_{\rm total}} + U_C = U_R + U_{\rm rad} + U_C, \tag{12}$$

even though the radiated energy $U_{\rm rad}$ is negligible in an "ordinary" R-C circuit.

In sum, when nominal resistance $R \to 0$, radiation becomes important, and the radiated energy $U_{\rm rad}$ approaches $CV^2/2$ (and energy is still conserved).

[4]This case is unusual in that for any small but finite resistance R, eqs. (7) and (9) still hold, so these equations hold if we were to consider that "zero resistance" means the limit of small resistance as this goes to zero.

In practice, all batteries have nonzero internal resistance, so even if superconducting wires were used there would be no paradox. However, if the battery were replaced by a charged capacitor, connected to an initially uncharged one via superconducting wires, the paradox would be more dramatic, as discussed in [1].

4 Energy Efficiency (added Oct. 16, 2020)

If the battery acts for only a finite time t, the charge accumulated on the capacitor follows from eq. (3) as,

$$Q(t) = \int_0^t I\,dt = \frac{V}{R}\int_0^t e^{-t'/RC}\,dt' = CV(1 - e^{-t/RC}), \tag{13}$$

and the stored electrical energy is,

$$U_{\rm cap}(t) = \frac{Q^2(t)}{2C} = \frac{CV^2}{2}(1 - e^{-t/RC})^2. \tag{14}$$

Meanwhile, the battery expended energy, recalling eq. (5),

$$\Delta U_{\rm batt}(t) = \int_0^t VI\,dt' = CV^2(1 - e^{-t/RC}).,\tag{15}$$

The energy efficiency of this process is,

$$\epsilon(t) = \frac{U_{\rm cap}(t)}{\Delta U_{\rm batt}(t)} = \frac{1 - e^{-t/RC}}{2} < \frac{1}{2} \qquad \text{(constant source voltage)}. \tag{16}$$

In contrast, if the battery were replaced by a constant-current source (for example, a van de Graaff generator [6], or, for short times, a photocell [7, 8]) of strength I, then the charge on the capacitor is $Q(t) = It$, the energy stored in the capacitor is,

$$U_{\rm cap}(t) = \frac{Q^2(t)}{2C} = \frac{I^2t^2}{2C}\,, \tag{17}$$

the energy dissipated in the resistance R is,[5]

$$\Delta U_R(t) = I^2 R\,t, \tag{18}$$

so the current source expended energy $\Delta U_{\rm source}(t) = U_{\rm cap}(t) + \Delta U_R(t)$, and the energy efficiency of the charging process is,

$$\epsilon(t) = \frac{U_{\rm cap}(t)}{\Delta U_{\rm source}(t)} = \frac{1}{1 + 2RC/t} \qquad \text{(constant source current)}, \tag{19}$$

which can approach unity for small R and/or large t. Indeed, a constant-current source (which corresponds to a linearly rising source voltage, $IR + I\,t/c$, is the most efficient source (in the sense of energy efficiency), as shown in [9]. See also [10].

[5]Any internal, series resistance of the current source is included in the resistance R.

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The comments on this paper by Oven, appended to the above link, seem misguided to the present author.

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