Canonical Angular Momentum of a Solenoid Field

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1 Problem

Consider a particle with charge e and momentum $\mathbf{P} = \mathbf{P}_z + \mathbf{P}_\perp$ ($P_\perp \neq 0$) that is moving on average in the z direction inside a solenoid magnet whose symmetry axis is the z axis and whose magnetic field strength is B_z. Inside the solenoid, the particle's trajectory is a helix of radius R, whose center is at distance R_0 from the magnet axis.

The longitudinal momentum P_z is so large that when the particle reaches the end of the solenoid coil, it exits the field with little change in its transverse coordinates. This behavior is far from the adiabatic limit in which the trajectory spirals around a field line.

When the particle exits the solenoid, the radial component of the magnetic "fringe" field exerts azimuthal forces on the particle, and, in general, leaves it with a nonzero azimuthal momentum, P_ϕ. Deduce a condition on the motion of the particle when within the solenoid, *i.e.*, on $R,\ R_0,\ P_z,\ P_\perp$, and B_z, such that the azimuthal momentum vanishes as the particle leaves the magnetic field region. Your result should be independent of the azimuthal phase of the trajectory when it reaches the end of the solenoid coil.

Hint: Consider the canonical momentum and/or angular momentum.

2 Solution

The key to this problem is conservation of canonical momentum, $\mathbf{P} + e\mathbf{A}/c$, where $\mathbf{A}$ is the vector potential (in Gaussian units).

It turns out to be even more effective to consider the canonical angular momentum, which is $\mathbf{L} = \mathbf{r} \times (\mathbf{P} + e\mathbf{A}/c)$.

We want $P_\phi = 0$ outside the magnet. This implies $L_z = rP_\phi = 0$ also. Therefore, we need $r(P_\phi + eA_\phi/c) = 0$ inside the magnet.

A solenoid magnet with field B_z has vector potential $A_\phi = rB_z/2$. To see this, recall that the integral of the vector potential around a loop is equal to the magnetic flux through the loop: $2\pi r A_\phi = \pi r^2 B_z$.

For a particle with average momentum in the z direction, its trajectory inside the magnet is a helix whose center is at some radius R_G (called R_0 in the statement of the problem) from the magnetic axis. The radius R_B (called R in the statement of the problem) of the helix can be obtained from $F = ma$:

$$\frac{mv_\perp^2}{R_B} = e\frac{v_\perp}{c}B_z, \tag{1}$$

so

$$R_B = \frac{cP_\perp}{eB_z}\,. \tag{2}$$

The direction of rotation around the helix is in the $-z$ direction (Lenz' law).

Since the canonical angular momentum is a constant of the motion, we can evaluate it at any convenient point on the particle's trajectory. In particular, we consider the point at which the trajectory is closest to the magnetic axis. As shown in Fig. 1, this point obeys $r = R_G - R_B$, and so

$$L_z = (R_G - R_B)\, P_\perp + \frac{eB_z}{2c}\,(R_G - R_B)^2 = \left(R_G^2 - R_B^2\right)\frac{eB_z}{2c}. \qquad (3)$$

Note that $R_G^2 - R_B^2$ is the product of the closest and farthest distances between the trajectory and the magnetic axis.

Hence, the canonical angular momentum vanishes for motion in a solenoid field if and only if $R_G = R_B$, *i.e.*, if and only if the particle's trajectory passes through the magnetic axis.

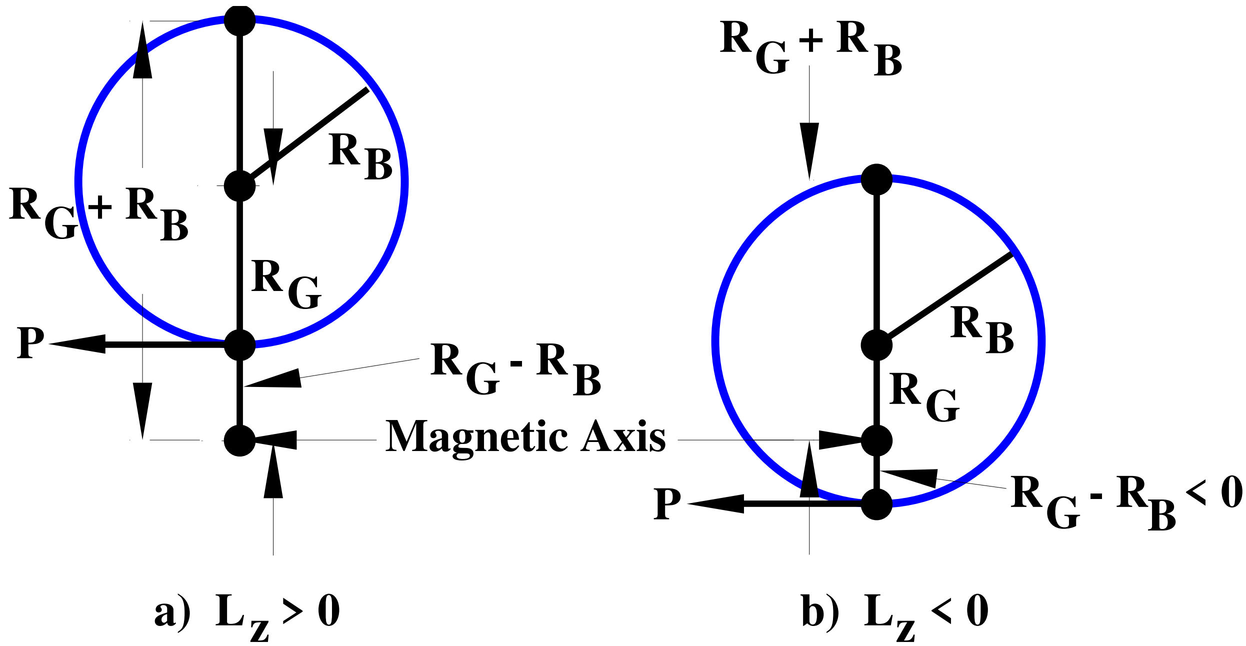

Figure 1: The projection onto a plane perpendicular to the magnetic axis of the helical trajectory a charge particle of transverse momentum P. The magnetic field B_z is out of the paper, so the rotation of the helix is clockwise for a positively charged particle. a) The trajectory does not contain the magnetic axis, and $L_z > 0$. b) The trajectory contains the magnetic axis, and $L_z < 0$.

We also see that if the trajectory does not contain the magnetic axis, the canonical angular momentum is positive; while if the trajectory contains the magnetic axis, the canonical angular momentum is negative.