Rotating Water Bucket with Lid

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(March 12, 2016)

1 Problem

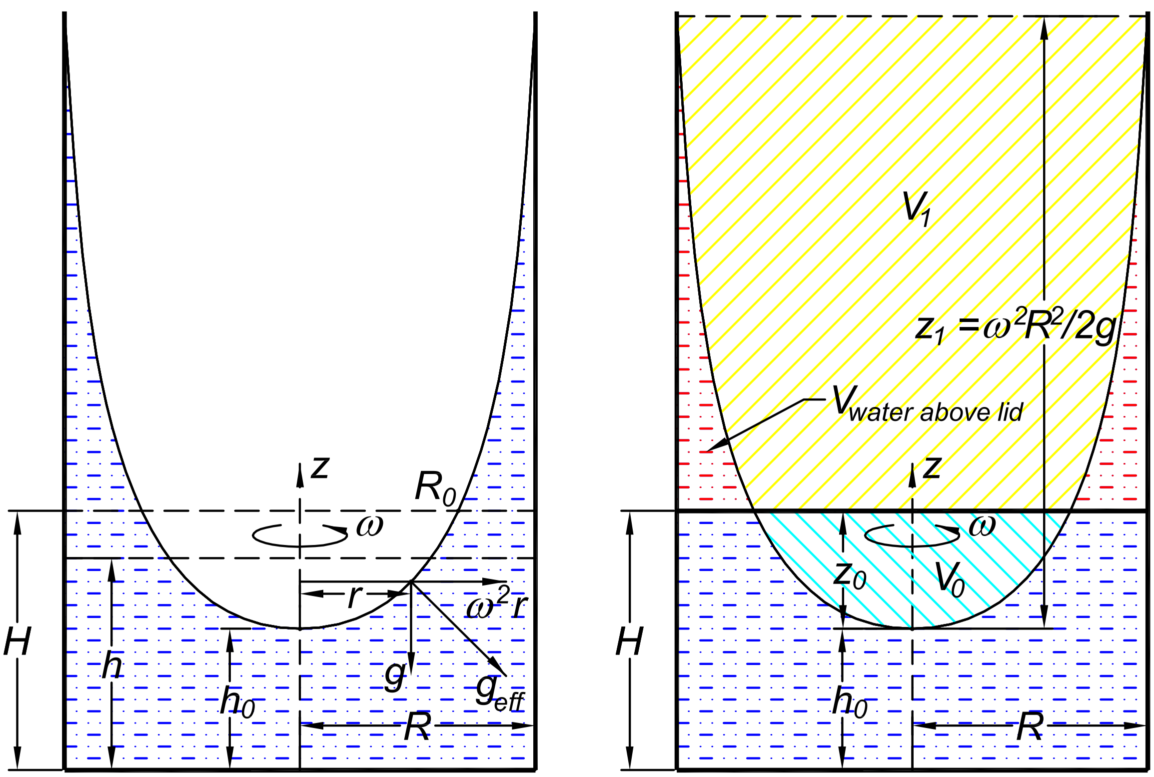
What is the force of water on the lid of a bucket that rotates about its vertical axis with angular velocity ω, if the water is in contact with the lid?[1] You may ignore air pressure.

This problem was suggested by Johann Otto.

2 Solution

2.1 Shape of the Water's Surface

To determine the shape of the surface of the water when the bucket is in steady rotation at angular velocity ω about the vertical (z) axis, we first consider a bucket with no lid, as sketched on the left below.



In the rotating frame, a mass element δm (at rest in the rotating frame) at distance r from the axis experiences a centrifugal force $\delta m \omega^2 r\,\hat{\mathbf{r}}$, in addition to the gravitational force $-\delta m g\,\hat{\mathbf{z}}$, where g is the acceleration due to gravity (in a nonrotating frame). We can regard

[1]Aspects of this problem were famously considered by Newton in his *Principia*, p. 81 of [1].

these two forces as components of an effective gravitational force, whose effective acceleration (in the rotating frame) is,

$$\mathbf{g}_{\text{eff}} = m\omega^2 r\,\hat{\mathbf{r}} - g\,\hat{\mathbf{z}}. \tag{1}$$

If the mass element lie on the surface of the water, the force due to the pressure of the surrounding water is equal and opposite to the force of the effective gravity, and the surface of the water is perpendicular to this vector force. That is, the slope of the water's surface is related by,

$$\frac{dz}{dr} = \frac{\omega^2 r}{g}, \qquad z = h_0 + \frac{\omega^2 r^2}{2g}. \tag{2}$$

2.2 Analysis in the Lab Frame

Once we know the shape (2) of the water's surface, we can make an analysis in lab frame by noting that if a lid is added to the bucket at height H above its base, and the water above the lid removed, the force on the lid by the water below it must equal the weight of the water removed (if the lid is not to move),

$$F_{\text{lid}} = \rho g V_{\text{water above lid}}, \tag{3}$$

where ρ is the density of water.

To compute the weight, we need to know the volume of the water above height H, supposing that when the bucket has no rotation the water surface is at height h. For this, we note that the volume of the air in the bucket with lid is,

$$V_{\text{air}} = \pi R^2(H - h), \tag{4}$$

where R is the radius of the bucket, and that this is also the volume of the (inverted) paraboloid whose apex is at height h_0 when the bucket has steady rotation at angular velocity ω. The volume of a paraboloid $z = \omega^2 r^2/2g$ of height $z_0 = H - h_0$ above its apex is given by,

$$V_{\text{paraboloid}}(z_0) = \int_0^{z_0} \pi r^2\,dz = \pi\int_0^{z_0} \frac{2gz}{\omega^2}\,dz = \frac{\pi g z_0^2}{\omega^2}. \tag{5}$$

Equating the two volumes, we find,

$$z_0^2 = \frac{\omega^2 R^2(H - h)}{g}, \qquad z_0 = \omega R\sqrt{\frac{H - h}{g}}, \tag{6}$$

such that the "base" of the inverted paraboloid has radius R_0 related by,

$$z_0 = \frac{\omega^2 R_0^2}{2g}, \qquad R_0 = \frac{\sqrt{2gz_0}}{\omega} = \sqrt{\frac{2R}{\omega}}\sqrt[4]{g(H - h)}, \tag{7}$$

and also that $h_0 = H - z_0$.

Now, if there were no lid, the water in the rotating bucket would reach height $h_0 + z_1$ where $z_1 = \omega^2 R^2/2g$. The volume of water above height H of the lid is, referring to the right figure on p. 1,

$$\begin{aligned}
V_{\text{water above lid}} &= \pi R^2(z_1 + h_0 - H) - [V_{\text{paraboloid}}(z_1) - V_{\text{paraboloid}}(z_0)] \\
&= \pi R^2(z_1 - z_0) - \frac{\pi g z_1^2}{\omega^2} + \frac{\pi g z_0^2}{\omega^2} \\
&= \frac{\pi\omega^2 R^4}{2g} - \frac{\pi\omega^2 R^2 R_0^2}{2g} - \frac{\pi\omega^2 R^4}{4g} + \frac{\pi\omega^2 R_0^4}{4g} = \frac{\pi\omega^2}{4g}(R^4 - 2R^2R_0^2 + R_0^4) \\
&= \frac{\pi\omega^2}{4g}(R^2 - R_0^2)^2 = \frac{\pi\omega^2 R^2}{4g}\left(R - \frac{2}{\omega}\sqrt{g(H-h)}\right)^2. \qquad (8)
\end{aligned}$$

The force on the lid is then,

$$F_{\text{lid}} = \rho g V_{\text{water above lid}} = \frac{\pi\rho\omega^2 R^2}{4}\left(R - \frac{2}{\omega}\sqrt{g(H-h)}\right)^2. \qquad (9)$$

2.3 Analysis in the Rotating Frame via Bernoulli's Equation

In the steady state, the water in the bucket is at rest in a frame that rotates with angular velocity ω about the axis of the bucket, such that in this frame Bernoulli's equation has the form (see, for example, sec. 2.3 of [2]),

$$P(r,z) + \rho g z - \frac{\rho\omega^2 r^2}{2} = \text{constant} = \rho g h_0, \qquad (10)$$

taking the pressure to be zero at the intercept of the surface of the water with the z-axis.

Then, the pressure of the water on the lid at height H, for $R_0 < r < R$ is,

$$P(r,H) = \frac{\rho\omega^2 r^2}{2} - \rho g(H - h_0) = \frac{\rho\omega^2 r^2}{2} - \rho g z_0. \qquad (11)$$

The force of the water on the lid is,

$$\begin{aligned}
F_{\text{lid}} &= 2\pi\int_{R_0}^{R} P(r,H)\, r\, dr = 2\pi\int_{R_0}^{R}\left(\frac{\rho\omega^2 r^2}{2} - \rho g z_0\right) r\, dr \\
&= \frac{\pi\rho\omega^2}{4}(R^4 - R_0^4) - \pi\rho g\frac{\omega^2 R_0^2}{2g}(R^2 - R_0^2) = \frac{\pi\rho\omega^2}{4}(R^4 - 2R^2R_0^2 + R_0^4) \\
&= \frac{\pi\rho\omega^2}{4}(R^2 - R_0^2)^2 = \frac{\pi\rho\omega^2 R^2}{4}\left(R - \frac{2}{\omega}\sqrt{g(H-h)}\right)^2, \qquad (12)
\end{aligned}$$

as previously found in eq. (9).

References

[1] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686),
http://kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf

[2] K.T. McDonald, *Bernoulli's Equation for a Rotating Fluid*, (Nov. 29, 2012),
http://kirkmcd.princeton.edu/examples/bernoulli_rot.pdf