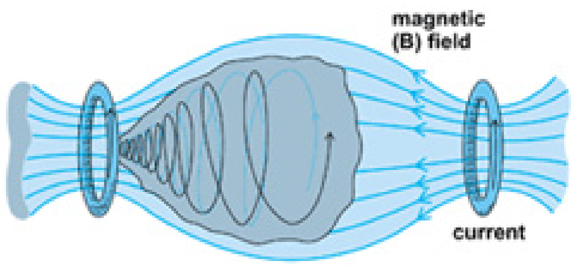
An Electric Bottle:
Charged Particle Orbiting a Charged Needle

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(May 1, 2013; updated November 9, 2016)

1 Problem

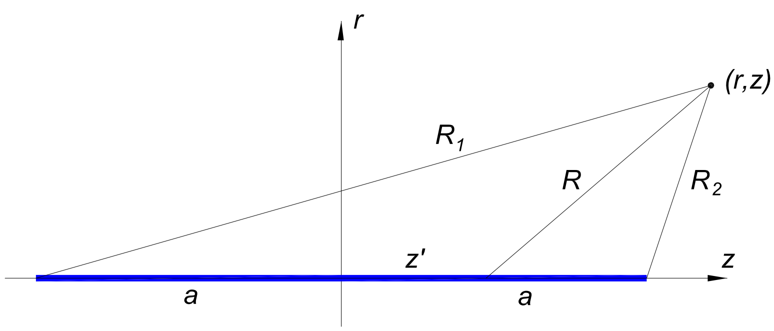
A "magnetic bottle" is a quasisolenoidal magnetic field, perhaps created by a pair of coils as in the figure below, such that a charged particle of sufficiently low velocity is "trapped" in the region between the coils, which act as "magnetic mirrors". See, for example, sec. 12.5 of [1].



Show that a charged particle orbiting a charged needle with sufficiently low velocity is similarly "trapped" in an "electric bottle". A delightful demonstration of this effect in the NASA Space Station is at [2].[1]

2 Solution

We take the needle to be a uniform line of length $2a$ and total charge Q, centered on the origin and along the z-axis of a cylindrical coordinate system (r, θ, z).



The electric scalar potential V at a point (r, θ, z) is, in Gaussian units,

$$V(r,z) = \int_{-a}^{a} \frac{Q/2a}{R}\, dz' = \frac{Q}{2a}\int_{-a}^{a} \frac{dz'}{\sqrt{r^2+(z-z')^2}} = -\frac{Q}{2a}\ln\left[z-z'+\sqrt{r^2+(z-z')^2}\right]_{-a}^{a}$$

[1]A related example of a charged needle inside a coaxial conductor cylinder is discussed in [3]. Such traps were first considered by Kingdon in 1923 [4]. A trap with a potential $U(r,\theta,z) = A(z^2 - r^2/2 + B\ln r)$ was discussed in [5], and is now commercialized as the Orbitrap [6, 7].

$$= \frac{Q}{2a}\ln\frac{z+a+R_1}{z-a+R_2}, \qquad \text{where} \qquad R_{1,2} = \sqrt{r^2+(z\pm a)^2}. \tag{1}$$

If we write,

$$u = \frac{R_1+R_2}{2}, \qquad v = \frac{R_1-R_2}{2}, \qquad \text{with} \qquad -\infty < u < \infty, \ -a \le v \le a, \tag{2}$$

then surfaces of constant u are prolate spheroids, surfaces of constant v are hyperboloids,

$$R_1 = u+v, \qquad R_2 = u-v, \qquad az = uv, \tag{3}$$

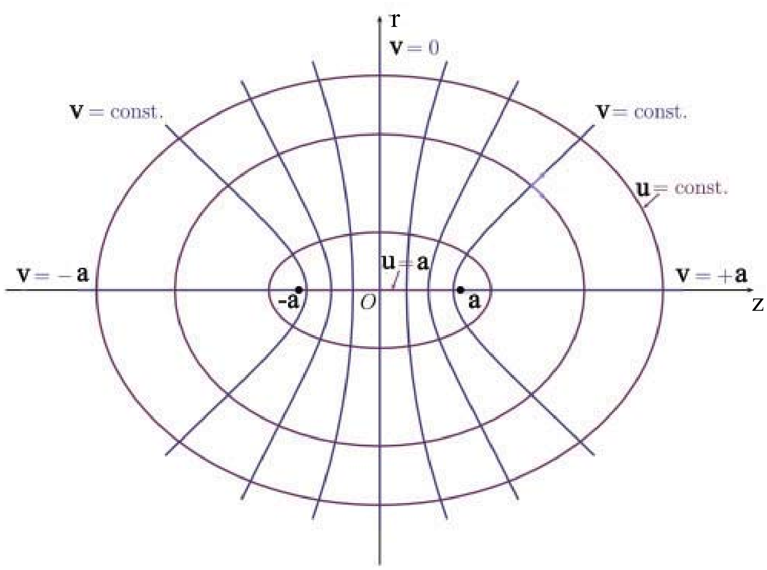
and,

$$\frac{z+a+R_1}{z-a+R_2} = \frac{uv+a^2+ua+va}{uv-a^2+ua-va} = \frac{(u+a)(v+a)}{(u-a)(v+a)} = \frac{u+a}{u-a}, \tag{4}$$

such that the electric potential is constant on surfaces of constant u,

$$V = \frac{Q}{2a}\ln\frac{u+a}{u-a}, \tag{5}$$

and the electric field lines lie on surfaces of constant v with no azimuthal component E_θ.



The coordinates (u, θ, v) form a prolate spheroidal coordinate system.

The electric field in the symmetry plane ($v = 0 = z$) has only a radial component,

$$E_r(r, z=0) = -\frac{\partial V(r, z=0)}{\partial r} = \frac{Q}{2a}\frac{\partial}{\partial r}\ln\frac{\sqrt{r^2+a^2}+a}{\sqrt{r^2+a^2}-a} = \frac{Q}{r\sqrt{r^2+a^2}}, \tag{6}$$

which falls off as $1/r^2$ at large radii.

An electric charge $-q$ with mass m can have a circular orbit in the plane $z = 0$ with any radius r_0 if its azimuthal speed $v_{0\theta} = r_0\omega_0$ is related by,

$$\omega_0^2 = \frac{qQ}{mr_0^2\sqrt{r_0^2+a^2}}, \qquad v_{0\theta}^2 = \frac{qQ}{m\sqrt{r_0^2+a^2}} = \frac{qQ}{am\sqrt{1+r_0^2/a^2}}. \tag{7}$$

It is useful to note the angular-momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ of the moving charge/mass obeys,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}) \times (F_r\,\hat{\mathbf{r}} + F_z\,\hat{\mathbf{z}}) = (zF_r - rF_z)\,\hat{\boldsymbol{\theta}}, \tag{8}$$

as the force $\mathbf{F} = -q\mathbf{E} = q\boldsymbol{\nabla}V(r,z)$ has no θ-component. Recalling that,

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\boldsymbol{\theta}, \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\,\mathbf{r}, \tag{9}$$

we have,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} = \frac{d}{dt}(r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}) = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{z}\,\hat{\mathbf{z}}, \tag{10}$$

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = m(r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}) \times (\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{z}\,\hat{\mathbf{z}}) = -mrz\dot{\theta}\,\hat{\mathbf{r}} + m(\dot{r}z - r\dot{z})\,\hat{\boldsymbol{\theta}} + mr^2\dot{\theta}\,\hat{\mathbf{z}}, \tag{11}$$

$$\frac{d\mathbf{L}}{dt} = -mz(2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\mathbf{r}} + m(\ddot{r}z + r\ddot{z} - rz\ddot{\theta})\,\hat{\boldsymbol{\theta}} + mr(2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\mathbf{z}}, \tag{12}$$

and hence from eq. (8), both the r and z-components of eq. (12) lead to,

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0, \qquad \text{and} \qquad L_z = mr^2\dot{\theta} = \text{constant}. \tag{13}$$

The z-component of the magnetic moment of the orbital motion of the charge q is,

$$\mu_z = \frac{I_\perp A_\perp}{c} = \frac{1}{c}\frac{q\dot{\theta}}{2\pi}\pi r^2 = \frac{qr^2\dot{\theta}}{2c} = \frac{q}{2mc}L_z. \tag{14}$$

That is, the magnetic moment μ_z is a constant of the motion whether or not the particle is trapped by the electric field.[2] This contrasts with the case of trapping in a magnetic bottle, where angular momentum L_z is not strictly conserved, and the magnetic moment is only an adiabatic invariant (for motion in magnetic fields that vary sufficiently slowly in space).[3]

We next consider perturbations about circular orbits in the plane $z = 0$, first for perturbations in this plane, and then (more interestingly) for perturbations perpendicular to it.

[2]The result (13), that the angular momentum L_z is constant, holds for any axially symmetric electric field (where the magnetic field is zero, which implies that the electric field is static). The mass m in eq. (14) is the "relativistic mass", $m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the rest mass, such that the magnetic moment is not strictly invariant for motion at high velocity v.

[3]Since motion in a magnetic field does not change the magnitude v of the charge's velocity $\mathbf{v}$, and hence the relativistic mass m is constant in a magnetic field, the magnetic moment can be an adiabatic invariant at high velocity, provided the spatial variation of the magnetic field is small enough that adiabatic invariance is a good approximation. However, the magnetic field must be more uniform for adiabatic invariance to hold at high velocity, compared to that sufficient for at low velocity.

2.1 Perturbed Motion in the Symmetry Plane

For motion in the plane $z = 0$, we have that,

$$\ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{L_z^2}{m^2 r^3} = F_r = -\frac{qQ}{r\sqrt{r^2 + a^2}}\,. \tag{15}$$

The equilibrium circular orbit has $\ddot{r} = 0$, and hence the equilibrium radius r_0 and angular velocity $\omega_0 = \dot{\theta}_0$ are related by,

$$\frac{L_z^2}{r_0^2} = \frac{qQm}{\sqrt{r_0^2 + a^2}}\,, \qquad \omega_0^2 = \frac{L_z^2}{m^2 r_0^4} = \frac{qQ}{m r_0^2 \sqrt{r_0^2 + a^2}}\,. \tag{16}$$

For small perturbations about the equilibrium orbit, we expand the radial equation of motion as,

$$\ddot{r} = \frac{L_z^2}{m^2 r^3} - \frac{qQ}{r\sqrt{r^2 + a^2}} \approx -\frac{qQ}{m r_0^2 \sqrt{r_0^2 + a^2}} \frac{r_0^2 + 2a^2}{r_0 + a^2}(r - r_0) = -\omega_0^2 \frac{r_0^2 + 2a^2}{r_0 + a^2}(r - r_0), \tag{17}$$

which has oscillatory solutions of the form,

$$r \approx r_0(1 + \epsilon \sin \omega t) \qquad \text{where} \qquad \omega^2 = \omega_0^2 \frac{r_0^2 + 2a^2}{r_0 + a^2} > \omega_0^2, \tag{18}$$

with period shorter than that of the equilibrium orbit. The angular velocity is given by,

$$\dot{\theta} = \frac{L_z}{mr^2} = \omega_0 \frac{r_0}{r^2} \approx \omega_0(1 - 2\epsilon \sin \omega t). \tag{19}$$

The perturbed orbits are ellipse-like with retrograde precession of the pericharge (to coin a phrase).

2.2 Perturbed Motion Perpendicular to the Symmetry Plane

We now consider motion that includes nonzero velocity $\dot{z}$ parallel to the axis of the charged rod. In particular, we consider motion with constant angular momentum $L_z = mr^2\dot{\theta}$, such that when $z = 0$, then $r = r_0$, $\dot{r} = 0$, $r\dot{\theta} = v_{0\theta}$ according to eq. (16), $L_z = m r_0 v_{0\theta}$, and $\dot{z} = v_{0z}$. The subsequent motion is qualitatively helical, and the total energy U is conserved,

$$\begin{aligned} U &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{qQ}{2a} \ln \frac{u + a}{u - a} = \frac{m}{2}\left(\dot{r}^2 + \frac{L_z^2}{m^2 r^2} + \dot{z}^2\right) - \frac{qQ}{2a} \ln \frac{u + a}{u - a} \\ &= \frac{m}{2}\left(\frac{L_z^2}{m^2 r_0^2} + v_{0z}^2\right) - \frac{qQ}{2a} \ln \frac{u_0 + a}{u_0 - a}\,, \end{aligned} \tag{20}$$

where $u_0 = \sqrt{r_0^2 + a^2}$, recalling the electric potential (5) and that,

$$u = \frac{\sqrt{r^2 + (z + a)^2} + \sqrt{r^2 + (z - a)^2}}{2}\,. \tag{21}$$

An interesting question is whether there exists a maximal value z_1 for the helical motion, such that the charge $-q$ is "trapped" in an "electric bottle", oscillating axially between "turning points" $\pm z_1$. The video [2] suggests that such "trapping" is possible for some values of v_{0z}.

If the turning point z_1 of the motion exists, then here $r = r_1$, $\dot{r} = 0$, $\dot{z} = 0$, and the energy equation (20) becomes,

$$\frac{L_z^2}{m^2 r_1^2} - \frac{qQ}{am} \ln \frac{u_1 + a}{u_1 - a} = \frac{L_z^2}{m^2 r_0^2} + v_{0z}^2 - \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a} \,. \tag{22}$$

The value of z_1 increases with increasing v_{0z}, and is infinite for the maximum axial speed $v_{0z,\max}$ for which a turning point exists. At infinite z_1 the electrical potential energy is zero, and eq. (22) becomes,

$$\frac{L_z^2}{m^2 r_1^2} = \frac{L_z^2}{m^2 r_0^2} + v_{0z,\max}^2 - \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a} \,. \tag{23}$$

Since $r_1 < r_0$ we have, recalling eq. (7),

$$v_{0z,\max}^2 > \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a} = v_{0\theta}^2 \sqrt{1 + \frac{r_0^2}{a^2}} \ln \frac{\sqrt{r_0^2 + a^2} + a}{\sqrt{r_0^2 + a^2} - a} \approx 2 v_{0\theta}^2 \ln \frac{2a}{r_0} \,, \tag{24}$$

where the approximation holds for $r_0 \ll a$. Thus, turning points $\pm z_1$ exist; for $r_0 \ll a$, the ratio $v_{0z,\max}/v_{0\theta}$ can be large compared to unity, and the case $v_{0z,\max} = v_{0\theta}$ holds for $r_0 \approx a\sqrt{e}/2 \approx 0.75a$. *The latter result is qualitatively consistent with the NASA video [2].*

2.2.1 Frequency of Small Axial Oscillations

Although the main interest in this problem is the existence of large-amplitude axial perturbations to the orbital motion, we also consider small axial perturbations for which an approximate calculation of the frequency can be given.

The axial equation of motion is,

$$\begin{aligned} m\ddot{z} &= -qE_x = q\frac{\partial V}{\partial z} = \frac{qQ}{2a}\frac{\partial}{\partial z} \ln \frac{u+a}{u-a} = -\frac{qQ}{u^2 - a^2}\frac{\partial u}{\partial z} \\ &= -\frac{qQ}{2(u^2 - a^2)} \left(\frac{z+a}{\sqrt{r^2 + (z+a)^2}} + \frac{z-a}{\sqrt{r^2 + (z-a)^2}} \right) . \end{aligned} \tag{25}$$

For $|z| \ll \sqrt{r_0^2 + a^2}$ we approximate,

$$u = \frac{\sqrt{r^2 + (z+a)^2} + \sqrt{r^2 + (z-a)^2}}{2} \approx \sqrt{r_0^2 + a^2}, \qquad \text{so} \qquad u^2 - a^2 \approx r_0^2, \tag{26}$$

and,

$$\begin{aligned} &\frac{z+a}{\sqrt{r^2 + (z+a)^2}} + \frac{z-a}{\sqrt{r^2 + (z-a)^2}} \\ &\approx \frac{1}{\sqrt{r_0^2 + a^2}} \left[(z+a)\left(1 - \frac{za}{r_0^2 + a^2}\right) + (z-a)\left(1 + \frac{za}{r_0^2 + a^2}\right) \right] = \frac{2 r_0^2 z}{(r_0^2 + a^2)^{3/2}} \,. \end{aligned} \tag{27}$$

Hence,

$$\ddot{z} \approx -\frac{qQz}{mr_0^2\sqrt{r_0^2+a^2}}\frac{r_0^2}{r_0^2+a^2} = -\omega_0^2\frac{r_0^2}{r_0^2+a^2}\,z, \tag{28}$$

recalling eq. (7). Thus, for $|z| \ll \sqrt{r_0^2+a^2}$, we have simple harmonic motion with angular frequency ω_z given by,

$$\omega_z^2 = \omega_0^2\frac{r_0^2}{r_0^2+a^2}\,. \tag{29}$$

For equilibrium orbits with $r_0 \gg a$, where the charged needle appears to be a point charge, we have that $\omega_z \approx \omega_0$ as expected for small oscillations, while for small r_0 the frequency of small axial oscillations is smaller than the equilibrium orbital frequency ω_0; *the charge $-q$ can complete several spiral turns per each cycle of axial oscillation, as seen in the video [2].*

References

[1] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[2] D.R. Pettit, *Science off the Sphere: Knitting Needles* (Feb. 6, 2012),
http://www.nasa.gov/multimedia/videogallery/index.html?media_id=131554451
https://www.youtube.com/watch?v=qHrBhgwq__Q
http://kirkmcd.princeton.edu/examples/EM/nasa_needle.mp4

[3] T. Biewer, D. Alexander and S. Robertson, *Electrostatic orrery for celestial mechanics*, Am. J. Phys. **62**, 821 (1994), http://kirkmcd.princeton.edu/examples/EM/biewer_ajp_62_821_94.pdf

[4] K.H. Kingdon, *A Method for the Neutralization of Electron Space Charge by Positive Ionization at Very Low Gas Pressures*, Phys. Rev. **21**, 408 (1923),
http://kirkmcd.princeton.edu/examples/EM/kingdon_pr_21_408_23.pdf

[5] R.D. Knight, *Storage of ions from laser-produced plasmas*, Appl. Phys. Lett. **38**, 221 (1981), http://kirkmcd.princeton.edu/examples/EM/knight_apl_38_221_81.pdf

[6] A. Makarov, *Electrostatic Axially Harmonic Orbital Trapping: A High-Performance Technique of Mass Analysis*, Anal. Chem. **72**, 1156 (2000),
http://kirkmcd.princeton.edu/examples/EM/makarov_ac_72_1156_00.pdf

[7] R.A. Zubarev and A. Makarov, *Orbitrap Mass Spectrometry*, Anal. Chem. **85**, 5288 (2013), http://kirkmcd.princeton.edu/examples/EM/zubarev_ac_85_5288_13.pdf