Bernoulli's Equation for a Rotating Fluid

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1 Problem

Deduce the pressure inside a non-viscous, incompressible fluid that rotates "rigidly" about a vertical axis with angular velocity ω, and discuss how this may be related to a version of Bernoulli's equation.

2 Solution

2.1 Bernoulli's Equation in the Lab Frame

If we can ignore viscous energy dissipation in the (incompressible) fluid, and its rotational motion is steady, then Bernoulli's equation holds in the lab frame,[1] such that,

$$P(r, \phi, z) + \frac{\rho v^2}{2} + \rho g h = \text{constant}, \tag{1}$$

along a streamline, where $P(r, \phi, z)$ is the internal fluid pressure in cylindrical coordinates with the fluid rotation about the (vertical) z-axis, ρ is the mass density of the fluid, v is its local velocity, and g is the acceleration due to gravity. For "rigid" rotation of the fluid about the z-axis, the streamlines are simply circles of constant r and z, and Bernoulli's equation (1) tells us only that the pressure is constant around any such circle, but it does not provide the relation between the pressure on different circles $(r, z) = \text{constant}$.

2.2 Newtonian Analysis of a Fluid Element

More information about the pressure can be obtained by consideration of the forces on a fluid element.

Such elements move in circles of radius r with radial acceleration $a_r = -\omega^2 r$, and have constant z. The fluid is axially symmetric, so the pressure is independent of angle ϕ for a given r and z, *i.e.*, $P = P(r, z)$. The vertical force on an element of height dz and horizontal area dA vanishes, so that,

$$[P(r, z) - P(r, z + dz) - \rho g\, dz]dA = 0, \qquad \frac{dP}{dz} + \rho g = 0, \qquad P(r, z) + \rho g z = \text{constant}. \tag{2}$$

The radial equation of motion of an element or radial extent dr and area dA in the ϕ-z plane is,

$$[P(r, z) - P(r + rd, z)]dA = -\rho\, dA\, dr\, \omega^2 r, \quad \frac{dP}{dr} = \rho\omega^2 r, \quad P(r, z) - \frac{\rho\omega^2 r^2}{2} = \text{constant}. \tag{3}$$

[1] An elegant derivation of Bernoulli's equation is given in sec. 5 of [1], starting from Euler's equation (sec. 2).

The pressure is higher at larger radii. Combining eqs. (2)-(3) we find that,

$$P(r,z) - \frac{\rho\omega^2 r^2}{2} + \rho g z = \text{constant}, \tag{4}$$

which has the form of the usual Bernoulli's equation, but with the opposite sign of the term $\rho v^2/2 = \rho\omega^2 r^2/2$.

If the rotating liquid has a free upper surface where the external pressure is P_0, eq. (4) for surface coordinates r_s and z_s is,

$$P(r_s, z_s) - \frac{\rho\omega^2 r_s^2}{2} + \rho g z_s = \text{constant}, \tag{5}$$

and the surface has the parabolic form,

$$z_s = z_0 + \frac{\omega^2 r_s^2}{2g}\,. \tag{6}$$

If the rotating liquid is in a horizontal vial with a (vertical) free surface at r_0, the fluid pressure at larger radii inside the vial has the form,

$$P(r) = P_0 + \frac{\rho\omega^2(r^2 - r_0^2)}{2}\,, \tag{7}$$

neglecting the small variation of pressure with height.

2.3 Bernoulli's Equation in the Rotating Frame

The fluid is a rest in the frame that rotates about the z-axis with angular velocity ω, so we expect that a version of Bernoulli's law should hold in this frame.

Further, since the fluid is at rest we can say that any two points are connected by a "streamline", so we should obtain a relation for the pressure everywhere in the liquid.

The fluid has no kinetic energy in the rotating frame, but it appears to be subject to a radial centrifugal-force density,

$$f_r = \rho\omega^2 r, \tag{8}$$

that can be deduced from a potential energy density,

$$V_{\text{centrifugal}} = -\frac{\rho\omega^2 r^2}{2}\,, \tag{9}$$

according to $f_r = -dV/dr$. Recalling that Bernoulli's equation is an expression of conservation of energy in which a (gravitational or other) potential energy density appear as an additive term, the desired version of Bernoulli's equation in the rotating frame is,

$$P(r,z) + \rho g z + V(r) = P(r,z) + \rho g z - \frac{\rho\omega^2 r^2}{2} = \text{constant}, \tag{10}$$

as previously found in eq. (4) via lab-frame analysis of force densities.[2]

In this example, Bernoulli's equation is advantageously considered in two different frames, in both of which the fluid motion is steady. For an example in which Bernoulli's equation applies only in the one of two frames where the fluid motion is steady, see [2].

2.4 Demonstrations

2.4.1 Rotating Hatboxes

Two right-circular cylinders ("hatboxes") are rotated about a vertical axis halfway between them, as shown below. The orange balloon is filled with helium and the black balloon with argon. The air in the "hatboxes" is at rest in the rotating frame, and the pressure inside the boxes increase with distance from the axis of rotation. Hence, the helium balloon "floats" towards the axis while the argon balloon "sinks" away from the axis.[3]

2.4.2 Whirly Tube

When a corrugated plastic tube is whirled about, say, a vertical axis, holding the base vertical as shown below, air is sucked in the base and flows out the whirling end, leading to possible acoustic resonances [3].[4]

[2](Mar. 9, 2002) If the fluid is not at rest in the rotating frame, but has velocity $\mathbf{v}$ at some point along a streamline of steady flow in that frame, Bernoulli's equation generalizes from eq. (1) to the form,

$$P(r,z) + \frac{\rho v^2}{2} + \rho g z - \frac{\rho \omega^2 r^2}{2} = \text{constant}, \tag{11}$$

See, for example, eq. (11.14), p. 766, of [4].

[3]Thanks to Omelan Stryzak for this demonstration.

[4]See [5] for a video in which a whirling tube deflates a garbage bag.

The behavior of the whirling tubes depends on its having a relative velocity relative to the bulk air; it would not "work" if it were place inside the rotating "hatboxes" of the previous demonstration. The air flow is not steady in the lab frame, nor in the frame of the whirling tube, so Bernoulli's equation does not apply (although it is often argued that the relative velocity of the air and the whirling end of the tube "create" low pressure there which sucks air through the tube). As the tube rotates, the air inside the tube acquires the angular velocity of the tube and experiences centrifugal force which drives a flow from the base (at rest) to the whirling end. A lower-velocity return flow exists outside the tube, so that some, but not all, of the outside air also has the angular velocity of the tube.[5]

References

[1] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Butterworth-Heinemann, 1998), http://kirkmcd.princeton.edu/examples/fluids/landau_fluids_59.pdf

[2] K.T. McDonald, *Is Bernoulli's Equation Relativistically Invariant?* (Sept. 13, 2009), http://kirkmcd.princeton.edu/examples/bernoulli.pdf

[3] F.S. Crawford, *Singing Corrugate Pipes*, Am. J. Phys. **42**, 278 (1974), http://kirkmcd.princeton.edu/examples/mechanics/crawford_ajp_42_278_74.pdf

[4] F.M. White, *Fluid Mechanics*, 7th ed. (McGraw-Hill, 2011), http://kirkmcd.princeton.edu/examples/fluids/white_11.pdf

[5] S. Spangler, *Whirly - The Twirling Sound Hose*, http://www.stevespanglerscience.com/experiment/sound-hose

[5] A rotating observer of air that is at rest in the lab frame can apply Bernoulli's equation to the streamlines (circles about the axis) according to $P + \rho v^2/2 - \rho\omega^2 r^2/2 = P =$ constant the streamlines, which are circles about the axis. However, this relation does not relate the pressure on different streamlines, so has little content.