Levitating Beachballs

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Discuss the vertical and horizontal stability of a beachball levitated by a vertical jet of air. This demonstration is familiar to denizens of science museums and hardware stores.

As a complete solution is difficult, you may restrict your discussion to a simplified example. Consider a beachball of radius a in a jet such that the Reynolds number is a few hundred. Then, to a good approximation the air flow is incompressible and laminar. Viscous drag can be ignored. The high-speed-drag coefficient C_D can be taken as 2.

The diverging (but divergence-free) flow from the jet can be modeled by noting that the (vertical) momentum flux[1] in the jet is $\rho_{\rm air} v_z^2$. If the jet has some effective cone angle, its area expands with height z as z^2, where $z = 0$ is the position of the nozzle of the vertical jet. Thus, to conserve the momentum flux, $v_z(z) \propto 1/z$. The transverse profile of the jet, $v_z(r)$, may be taken as Gaussian (or parabolic) in radius r of a cylindrical coordinate system.

You may also make the unrealistic approximation that the air flow is unperturbed by the beachball.

2 Solution

(July 31, 2021) While the solution offered below is perhaps naïvely satisfying, it ignores viscosity (friction/kinetic-energy dissipation), which has a nontrivial effect in gas/liquid jets. As remarked by Weltner [2], the static pressure in the air jet is actually the same as outside the jet, and the observed "trapping" of a sphere in the jet is better thought of as due to the Coanda effect (that friction/viscosity leads to fluid flow lines being "attracted" to solid surfaces).[2] Indeed, the air jet need not be vertical to "trap" a sphere, which effect is not expected in the Bernoulli's-law analysis given below.

Unfortunately, consideration of the Coanda effect does not lead to such a simple analysis as that presented below.

Other discussions of this and related problems include [11]-[15].

2.1 Brief Discussion

The beachball is subject to three forces: gravity, high-speed drag and pressure-gradient forces. Gravity is downwards. The drag force is in the direction of the flow velocity $\mathbf{v}$, hence upwards and away from the axis of the vertical jet. The pressure-gradient force points in

[1] Momentum flux = momentum/area/time = momentum density × velocity.

[2] Coanda published his early efforts only as patents [3]-[7], which are now generally acknowledged as describing the "Coanda effect", as well as proposing the first (turboprop) jet engine. See also [8]-[10].

the direction of lower pressure, *i.e.*, the direction of higher v^2 according to Bernoulli's law; hence, downwards and towards the axis of the jet.

For levitation to occur, the high-speed drag force must be large enough to counteract gravity and the pressure-gradient force. If so, we anticipate that vertical motion in the vicinity of the equilibrium point is stable. The situation for horizontal motion is less clear,since the (large) drag force destabilizes the equilibrium.

In the model discussed below, we find that the equilibrium is stable against vertical perturbations if the jet is reasonably well collimated.

2.2 The Model

We work in cylindrical coordinates (r, z) with the z axis vertical.

In this problem, we are concerned with the flow on or near the z axis, where $v_r \ll v_z$. While a first approximation will suffice for v_r, we need a second approximation for v_z. To second order, the dependence of v_z on radius r can be approximated by a Gaussian, or by a parabola,

$$v_z \approx \frac{A}{z} e^{-r^2/2\beta^2 z^2} \approx \frac{A}{z}\left(1 - \frac{r^2}{2\beta^2 z^2}\right), \tag{1}$$

where A is a constant of dimensions length $\times$ velocity and β is the root-mean-square cone angle of the jet. For what it's worth, the jet momentum that crosses a surface of constant z each second is $J_z = \int 2\pi r dr\ \rho_a v_z^2(r, z) = \pi \rho_a A^2 \beta^2$ where ρ_a is the density of air.

We determine v_r by requiring that $\nabla \cdot \mathbf{v} = 0$, which holds assuming air to be incompressible. Thus,

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} = -\frac{\partial v_z}{\partial z} \approx \frac{A}{z^2}\left(1 - \frac{3r^2}{2\beta^2 z^2}\right), \tag{2}$$

which integrates to,

$$v_r \approx \frac{Ar}{2z^2}\left(1 - \frac{3r^2}{4\beta^2 z^2}\right). \tag{3}$$

Our model is a simplified version of Schlichting's (1933) solution for the laminar flow from a circular nozzle [1],

$$v_z = \frac{A}{z}\frac{1}{(1 + Br^2/z^2)^2} \approx \frac{A}{z}\left(1 - 2Br^2/z^2\right)^2, \tag{4}$$

$$v_r = \frac{Ar}{2z^2}\frac{1 - Br^2/z^2}{(1 + Br^2/z^2)^2} \approx \frac{Ar}{2z^2}\left(1 - 3Br^2/z^2\right)^2. \tag{5}$$

Our model agrees with the approximate form of Schlichting's on identifying $B = 1/4\beta^2$.

In the present problem, we only need the first approximation for the radial velocity profile,

$$v_r \approx \frac{Ar}{2z^2}. \tag{6}$$

2.3 The Three Forces on the Ball

2.3.1 Gravity

$$\mathbf{F}_g = -mg\hat{\mathbf{z}} = -\frac{4}{3}\pi a^3 \rho_b g \hat{\mathbf{z}}, \tag{7}$$

where ρ_b is the density of the ball.[3]

2.3.2 High-Speed Drag

$$\mathbf{F}_{\text{drag}} = \frac{C_D}{2}\rho_a \pi a^2 v^2 \hat{\mathbf{v}}, \tag{8}$$

where ρ_a is the density of the air and velocity $\mathbf{v}$ is evaluated at the center of the ball using the eqs. (1) and (6). We take $C_D/2 = 1$ in our regime.

2.3.3 Pressure-Gradient Effects

We again suppose that the ball does not perturb the pressure distribution. The net force in some direction $\mathbf{u}$ on the ball due to the pressure variation can be calculated using a spherical coordinate system centered on the ball, with angle θ measured with respect to the u-axis,

$$F_{\nabla P} = -a^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi \; P(a,\theta,\phi)\cos\theta. \tag{9}$$

We ignore the dependence of P on ϕ and approximate

$$P(a,\theta,\phi) \approx P(0) + P'(0) a\cos\theta + ... \tag{10}$$

where the derivative is with respect to $u = a\cos\theta$. Then,

$$F_{\nabla P} = -\frac{4}{3}\pi a^3 P' + ... \tag{11}$$

We relate pressure to velocity by Bernoulli's equation (ignoring the gravitational pressure difference),

$$P + \frac{1}{2}\rho_a v^2 = P_0, \tag{12}$$

where P_0 is the atmospheric pressure far from the jet. Hence,

$$F_{\nabla P,u} = \frac{2}{3}\pi a^3 \rho_a \frac{\partial v^2}{\partial u} + ... \tag{13}$$

[3]We neglect the buoyant force $4\pi a^3 \rho_b g\hat{\mathbf{z}}/3$ on the ball, supposing that $\rho_a \ll \rho_b$.

2.4 The Equilibrium Point

Clearly, an equilibrium point lies on the z axis above the jet, so we take $u = z$ and write,

$$F_z(z) = F_g + F_{\mathrm{drag},z} + F_{\nabla P,z} = -\frac{4}{3}\pi a^3 \rho_b g + \pi a^2 \rho_a v^2 + \frac{2}{3}\pi a^3 \rho_a \frac{dv^2}{dz} \tag{14}$$

Along the axis, the flow is given by (1) as $v^2 = A^2/z^2$, so,

$$F_z(z) = -\frac{4}{3}\pi a^3 \rho_b g + \frac{\pi a^2 A^2 \rho_a}{z^2} - \frac{4\pi a^3 A^2 \rho_a}{3z^3}. \tag{15}$$

The equilibrium height z_0 then satisfies,

$$\frac{1}{z_0^2}\left(1 - \frac{4a}{3z_0}\right) = \frac{4a\rho_b g}{3A^2\rho_a}. \tag{16}$$

The lefthand side of this equation must be positive, so the equilibrium height z_0 must be greater than $4a/3$ above the jet nozzle. If the ball is too heavy, it just falls onto the nozzle in our approximation. In practice, as the ball comes very close to the nozzle, the perturbations in the flow cannot be ignored, and an equilibrium might still be possible.

2.5 Vertical Oscillations about Equilibrium

As usual, we expand,

$$F_z(z) = F'(z_0)(z - z_0) + ... \tag{17}$$

If $F'(z_0)$ is negative then the vertical motion is stable, with oscillations about equilibrium at frequency $\omega = \sqrt{-F'(z_0)/m}$. From (15) we find,

$$F'(z_0) = -\frac{2\pi a^2 A^2 \rho_a}{z_0^3}\left(1 - \frac{2a}{z_0}\right). \tag{18}$$

It appears that the vertical equilibrium is stable only for $z_0 \gtrsim 2a$, a slightly stronger condition than (16) that the equilibrium exist.

2.6 Stability of Horizontal Motion

The horizontal force on the ball in the plane $z = z_0$ is,

$$F_r(r, z_0) = F_{\mathrm{drag},r} + F_{\nabla P,r} = \pi a^2 \rho_a v v_r + \frac{2}{3}\pi a^3 \rho_a \frac{\partial v^2}{\partial r}, \tag{19}$$

using (8) and (13). We are only interested in small perturbations away from the axis $r = 0$, so we keep terms in F_r only to order r/z, corresponding to order r^2/z^2 in v^2. Then,

$$v^2 = v_z^2 + v_r^2 \approx \frac{A^2}{z^2}\left(1 - \frac{r^2}{\beta^2 z^2} + \frac{r^2}{4z^2}\right), \tag{20}$$

$$\frac{\partial v^2}{\partial r} \approx -\frac{2A^2 r}{\beta^2 z^4}\left(1 - \frac{\beta^2}{4}\right), \tag{21}$$

from (1-6), and,

$$vv_r \approx \frac{A^2 r}{2z^3}. \tag{22}$$

Combining (19-22),

$$F_r(r, z_0) \approx -\frac{4\pi a^3 A^2 \rho_a r}{3\beta^2 z_0^4}\left(1 - \frac{\beta^2}{4} - \frac{3\beta^2 z_0}{8a}\right). \tag{23}$$

In our model, β is the cone angle of the jet, so $\beta \lesssim 1$; otherwise we could hardly speak of a jet. Hence, the term $\beta^2/4$ will not cause instability by itself. The term $3\beta^2 z_0/8a$ could lead to instability if $z_0 \gg a$, *i.e.*, for equilibrium points very far above the jet nozzle. Since βz_0 is the radius of the jet at height z_0, we must have $\beta z_0 \gtrsim a$ for the ball to be within the jet, so the model is meaningful. If we take $\beta z_0 \approx 2a$ as the working regime, then $3\beta^2 z_0/8a \approx 3\beta/4$. Furthermore, (23) indicates that there would be horizontal stability for any value of β less than 1, *i.e.*, for all reasonable jets.

The frequencies of the horizontal and vertical oscillations are typically not the same, and result in a "jumpy" appearance to the motion of a levitating beachball. For $z_0 \gg a$ and $\beta z_0 = ka$, the ratio of frequencies is,

$$\frac{\omega_{\rm horiz}}{\omega_{\rm vert}} \approx \frac{1}{k}\sqrt{\frac{2z_0}{3a}}. \tag{24}$$

This can be close to unity, for example, with $z_0 = 6a$ and $\beta z_0 = 2a$.

Among the possibilities for the orbit of the horizontal oscillations is a circle.

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