A magnetic field based on Ampère's force law

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Abstract

Ampère's force law for steady currents was not historically associated with a magnetic field, but it could have been. A magnetic field, inspired by work of Helmholtz in 1870, can be defined such that the double-differential form of Ampère's force law is a function of a double-differential of this field. We call this field the Ampère-Weber field, $\vec{\mathcal{B}}$, and show that its divergence is everywhere zero, as is that of the usual, but different magnetic field $\mathbf{B}$ of Maxwellian electrodynamics. The curl of the Ampère-Weber field is nonzero everywhere in static examples, in contrast to that of the usual magnetic field $\mathbf{B}$. We illustrate the field $\vec{\mathcal{B}}$ for three examples, which exhibit patterns of field lines quite different from those of the usual the magnetic field. As the Ampère-Weber field is based on Ampère's force law for steady currents, it does not extrapolate well to the Lorentz force on a moving charge in a magnetic field. That is, the Ampère-Weber field $\vec{\mathcal{B}}$, like Ampère's force law, is more of a curiosity than a viable alternative to the usual magnetic field $\mathbf{B}$. If the Ampère-Weber field had been invented in the mid 1800's, it would have been a distraction more than a step towards a generally valid electromagnetic field theory.

After a historical introduction in sec. 1, we discuss the Ampère-Weber field in sec. 2.

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1 Historical Background

1.1 Ampère

1.1.1 Ampère's Force Law

In 1820-1822, Ampère examined the force between two circuits, 1 and 2, carrying steady currents I_1 and I_2. He did not use vector notation, but his result on pp. 21-24 of [1] is equivalent to,

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\rm A}(\mathbf{x}_1, \mathbf{x}_2) = -\mathbf{F}_{\text{on }2}, \tag{1}$$

with,

$$d^2\mathbf{F}_{\rm A}(\mathbf{x}_1, \mathbf{x}_2) = -d^2\mathbf{F}_{\rm A}(\mathbf{x}_2, \mathbf{x}_1) = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2}\,, \tag{2}$$

where $d^2\mathbf{F}_{\rm A}(\mathbf{x}_1, \mathbf{x}_1)$ is the force on current element $I_1\, d\mathbf{l}_1$ at $\mathbf{x}_1$ due to current element $I_2\, d\mathbf{l}_2$ at $\mathbf{x}_2$, and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ [5]. We use SI units in this paper; μ_0 is the permeability of the vacuum [6].

The integrand $d^2\mathbf{F}_{\rm A}(\mathbf{x}_1, \mathbf{x}_2)$ of eq. (1) has the appeal that it changes sign if elements 1 and 2 are interchanged, and hence Ampère's force law for current elements obeys Newton's third law [12].

Ampère's force law (1) was generally accepted as the proper representation of (static) magnetic forces until around 1890 [13], when "electron theory" emerged from studies of electrical discharges in low-pressure gases, and Lorentz' generalization, Sec. 17 of [17], of the Biot-Savart force law replaced Ampère's force law.

1.1.2 Ampère's Circuital Law

In 1826, Ampère gave lectures that included discussion of the force on a magnetic pole due to an electric current, noting that the line integral of the tangential force around a closed loop is proportional to the electric current that passes through the loop, independent of the shape of the loop [18, 19]. This was a statement of what is now called "Ampère's (circuital) law" [20]. While Ampère did not consider the now-usual magnetic field $\mathbf{B}$, we note that the force on a magnetic pole p is $\mathbf{F} = p\,\mathbf{B}$, so his conclusion that $\oint \mathbf{F} \cdot d\mathbf{l} \propto I$, where I is the electric current through the loop of integration, implies also that $\oint \mathbf{B} \cdot d\mathbf{l} \propto I$.

Maxwell was the first to state Ampère's circuital law in terms of a magnetic field, via a verbal description on p. 56 of [23], where he deduced from Stokes' theorem that $\nabla \times \mathbf{H} = \mathbf{J}$. Maxwell's use of $\mathbf{H}$ rather than $\mathbf{B}$ was in the context of the relation $\mathbf{B} = \mu\mathbf{H}$ for linear media, his eq. (B), p. 53, where our μ is Maxwell's k_2. See also Art. 498 of [7], and pp. 372-373 of [24].

1.2 Biot and Savart

In 1825, Ampère noted [26] that for closed circuits eqs. (1)-(2) can be rewritten as,

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{B-S}}(\mathbf{x}_1, \mathbf{x}_2) = -\mathbf{F}_{\text{on }2}, \tag{3}$$

where,

$$d^2\mathbf{F}_{\text{B-S}}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mu_0}{4\pi} I_1 I_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2} = I_1\, d\mathbf{l}_1 \times \frac{\mu_0}{4\pi} \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{4}$$

in vector notation, such that the total force $\oint_1 \oint_2 d^2\mathbf{F}_{\text{B-S}}$ on closed circuit 1 due to closed circuit 2 is the same as with use of eq. (1). Ampère made very little comment on this result, other than noting that $d^2\mathbf{F}_{\text{B-S}}(\mathbf{x}_1, \mathbf{x}_2)$ does not obey Newton's third law, omitting that it was inspired by the magnetic force law of his rivals Biot and Savart [28]. As a consequence, the form (4) is generally attributed to Grassmann [15], as in [34], for example.

In retrospect, we see that eq. (4) lends itself to the interpretation that the force between closed circuits with steady currents can be written in terms of a magnetic field $\mathbf{B}_{\text{B-S}}$ as,

$$\mathbf{F}_{\text{on }1} = \oint_1 d\mathbf{F}_{\text{B-S}}(\mathbf{x}_1) = \oint_1 I_1(\mathbf{x}_1)\, d\mathbf{l}_1 \times \mathbf{B}_{\text{B-S}}(\mathbf{x}_1), \tag{5}$$

where,

$$\mathbf{B}_{\text{B-S}}(\mathbf{x}_1) = \frac{\mu_0}{4\pi} \oint_2 \frac{I_2(\mathbf{x}_2)\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{6}$$

with $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. Equations (5)-(6) are a factorization of the Biot-Savart force law (4), and both equations are called the Biot-Savart law in, for example, sec. 7-6, p. 125 of [35]. The earliest description of eq. (5) as the Biot-Savart law may be in sec. 2 of [36]. However, many authors call only eq. (6) the Biot-Savart law (while it is called Laplace's law in France); an early example is on p. 220 of [37]. Equation (5) is often called the Lorentz force law, although it was first stated by Maxwell, somewhat obscurely, as the third term in eqs. (12)-(14), p. 172 of [38], and more clearly in eq. (11), Art. 603 of [7].

The magnetic field $\mathbf{B}$ of Maxwellian electrodynamics is the time-dependent generalization of the Biot-Savart form (6).

1.3 Faraday

Ampère had no concept of a magnetic field, which originated with Faraday, inspired in part by patterns of iron filings on a sheet near a magnet [39]. Of particular interest here is Fig. 3 from Art. 3295 of [42], in which Faraday showed the pattern of iron filings in a plane containing the axis of a small dipole magnet, as shown in fig.1 below.

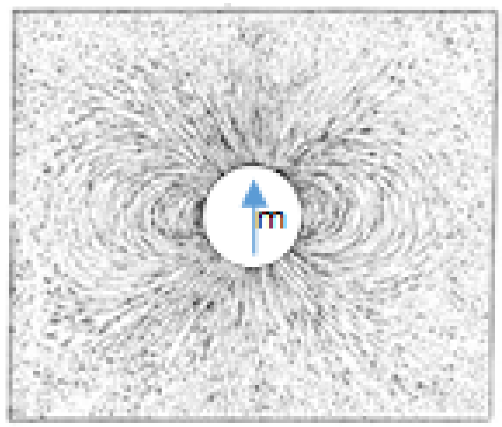

Fig. 1. The pattern of iron filings in a plane containing a magnetic moment $\mathbf{m}$. From [42].

This pattern corresponds to the lines of force of a magnetic dipole $\mathbf{m}$ on a hypothetical magnetic pole p as deduced by Poisson, eq. (9), p. 507 of [43] (1824),

$$\mathbf{F} = -p\,\boldsymbol{\nabla}\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{r^2} = p\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,. \tag{7}$$

where $\mathbf{r}$ is the vector from the center of the dipole $\mathbf{m}$ to the pole p. This was regarded by Poisson as an action-at-a-distance force, and he did not consider the possibility of a magnetic force field such as $\mathbf{B} = \mathbf{F}/p$ that existed in vacuum at points unoccupied by magnetic poles.

Our present view is that iron filings are not magnetic poles, but magnetic dipoles, which align themselves along the magnetic field $\mathbf{B}$.

1.4 Lord Kelvin

The first to adopt Faraday's concept of a magnetic field was Thomson (later Lord Kelvin), who discussed the magnetic field of a magnetic dipole $\mathbf{m}$ in sec. II of [44] (1846, age 22). However, he did not follow the path of Poisson to write $\mathbf{B} = -\boldsymbol{\nabla}(\mathbf{m}\cdot\hat{\mathbf{r}}/r^2$, but simply stated that,

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \text{where} \qquad \mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}\,, \tag{8}$$

in his eq. (II) where $\mathbf{B} = (X, Y, Z)$, and in his eq. (3) where $\mathbf{A} = (\alpha, \beta, \gamma)$. This is the first appearance of a vector potential in print [45]. Like Poisson, Thomson provided no figure, but gave a brief verbal description that suggests he was aware of the form, eq. (7), given by Poisson, which agrees with our eq. (8), assuming that $\mathbf{F} = p\mathbf{B}$.

1.5 Neumann

Meanwhile, in 1845, Neumann followed the examples of Lagrange, Laplace and Poisson in relating forces of gravity and electrostatics to (scalar) potentials, and sought a potential for Ampère's force law (2) between two (closed, steady) current loops.

For this, he noted that this force law can be rewritten in the as eq. (10) below, which permits us to write $\mathbf{F}_{\mathrm{on1}} = -\boldsymbol{\nabla} U$ where, U is the scalar potential (energy),

$$U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}\,, \tag{9}$$

eq. (9) above, given on p. 8 (also p. 67) of [27].

Neumann's argument was that for a element at $\mathbf{x}_1$, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\,dr = (df/dr)\,d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (4) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write,

$$\mathbf{F}_{\mathrm{on}\ 1} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\,\hat{\mathbf{r}} = -\mathbf{F}_{\mathrm{on}\ 2} = -\nabla U. \tag{10}$$

We now also write magnetic energy as,

$$U = I_1 I_2 M_{12} = I_1 \oint_1 d\mathbf{l}_1 \cdot \mathbf{A}_2 = I_1 \int d\mathbf{Area}_2 \cdot \boldsymbol{\nabla} \times \mathbf{A}_2 = I_1 \int d\mathbf{Area}_2 \cdot \mathbf{B}_2 = I_1 \Phi_{12}, \tag{11}$$

where M_{12} is the mutual inductance of circuits 1 and 2, Φ_{12} is the magnetic flux through circuit 1 due to the steady current I_2 in circuit 2, and,

$$\mathbf{A}_2 = \frac{\mu_0}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2}{r}\,, \tag{12}$$

such that Neumann is often credited for inventing the vector potential $\mathbf{A}$, although he appears not to have written our eq. (9) in any of the forms of eq. (11).

1.6 Weber

In 1846 Weber published a force law for moving charges, p. 327 of [49], p. 149 of [50], which he extraploated from Ampère's force law (1)-(2),

$$\mathbf{F}_{\text{Weber}} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\left[1 - \frac{1}{c^2}\left(\frac{\partial r}{\partial t}\right)^2 + \frac{r}{c^2}\frac{\partial^2 r}{\partial t^2}\right]\hat{\mathbf{r}} \tag{13}$$

where charge q_1 is at $\mathbf{x}_1$, charge q_2 is at $\mathbf{x}_2$, and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. However, Weber considered that his force law, like Ampère's, involved instantaneous action at a distance, and he did not make a connection between electromagnetism and light, regarding $1/c^2 = \epsilon_0\mu_0$ as related to electro- and magnetostatics. Also, like Ampère, Weber had no concept of a magnetic field.

1.7 Maxwell

Maxwell's discussions of Ampère's force law, Ampère's circuital law, and the Biot-Savart force law were mentioned at the ends of secs. 1.1.1 (see also [13]), 1.1.2 and 1.2 above, respectively.

1.7.1 Vector Potential

Maxwell may have been the next person after Thomson to publish a discussion of a vector potential, when in 1856, p. 63 of [23], he related the magnetic energy U_m of electric currents with density $\mathbf{J}$ to a vector potential $\mathbf{A}$ such that,

$$U_m = \int \frac{\mathbf{J}\cdot\mathbf{A}}{2}\, d\text{Vol}. \tag{14}$$

In [23], Maxwell considered that magnetic charges might exist, with density ρ_m, such that the magnetic field $\mathbf{B}$ obeyed $\nabla\cdot\mathbf{B} = \mu_0\rho_m$ and $\nabla\times\mathbf{A} \neq \mathbf{B}$. But, he did relate the electric field induced by time-dependent currents to the vector potential as $\mathbf{E}_{\text{induced}} = -\partial\mathbf{A}/\partial t$, on p. 64 of [23].

1.7.2 Lorentz Force

Maxwell wrote the "Lorentz" force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$, for charge q with velocity $\mathbf{v}$ in electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, as eq. (77), p. 342 of [51] (1861), eq. (D), p. 485 of [52] (1865), eq. (B), Art. 598 and eq. (1), Art. 599 of [7] (1873). However, these were little understood at the time.

1.8 Kirchhoff

Kirchhoff [53], using Weber's electrodynamics, deduced a wave equation for the current and charge on current elements (conductors) of small resistance, finding the wavespeed to be $c = 1/\sqrt{\epsilon_0 \mu_0}$, where the constants ϵ_0 and μ_0 can be determined from electro- and magnetostatic experiments (Weber and Kohlrausch (1856) [54]), and the value of c was close to the speed of light as then known. However, as Weber's electrodynamics was based on action-at-a distance, and was not a field theory in the sense of Faraday, Kirchhoff (like Weber) did not infer that light must be an electromagnetic phenomenon.

Kirchhoff's analysis on p. 530 of [53] involved a vector potential,

$$\mathbf{A}_{\rm W}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \cdot \hat{\mathbf{r}} \, \frac{\hat{\mathbf{r}}}{r} \, d{\rm Vol}', \tag{15}$$

with $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, that he attributed to Weber, who later transcribed Kirchhoff's paper into sec. I.1. of [56], with $\mathbf{A}_{\rm W}$ appearing on p. 578.

1.9 Helmholtz

In 1870, Helmholtz made a review of electrodynamics, and in eq. (1), p. 76 of [57] (see also [58]), he stated that a general form for the magnetic interaction energy (his P, but our U) of two current elements, which are part of closed circuits of steady currents, could be written as a combination of the forms he attributed to Neumann [27,64] and to Weber [49,65],

$$d^2U = \frac{\mu_0}{4\pi} \left(\frac{1+k}{2} \frac{I_1 \, d\mathbf{l}_1 \cdot I_2 \, d\mathbf{l}_2)}{r} + \frac{1-k}{2} \frac{(I_1 \, d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(I_2 \, d\mathbf{l}_2 \cdot \hat{\mathbf{r}})}{r} \right), \tag{16}$$

where $k = 1$ for Neumann's form and $k = -1$ for Weber's form [66]. Then, in eq. (1^a) he argued that the scalar U is related to a vector potential (his (U, V, W) but our $\mathbf{A}$) as $U = \int \mathbf{J} \cdot \mathbf{A} \, d{\rm Vol}/2$, noting that $I \, d\mathbf{l} \leftrightarrow \mathbf{J} \, d{\rm Vol}$, where $\mathbf{J}$ is the (steady) current density (which obeys $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$) [67], with,

$$\mathbf{A} = \frac{1+k}{2} \mathbf{A}_{\rm N} + \frac{1-k}{2} \mathbf{A}_{\rm W}, \tag{17}$$

where,

$$\mathbf{A}_{\rm N}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{r} \, d{\rm Vol}', \tag{18}$$

and $\mathbf{A}_W$ is given in eq. (15). However, Neumann never wrote the form called $\mathbf{A}_{\rm N}$ here, although this is the form Gauss claimed (1867) to have deduced in 1835 [45]. Both $\mathbf{A}_{\rm N}$ and $\mathbf{A}_{\rm W}$ lead to the same magnetic field, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_{\rm N} = \boldsymbol{\nabla} \times \mathbf{A}_{\rm W}$, which is an early example of gauge invariance [68].

2 The Ampère-Weber Field

We now consider the field,

$$\vec{\mathcal{B}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \oint I\, d\mathbf{l} \cdot \hat{\mathbf{r}}\, \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \cdot \hat{\mathbf{r}}\, \frac{\hat{\mathbf{r}}}{r^2}\, d\mathrm{Vol}' \tag{19}$$

which has the form of $\mathbf{A}_{\mathrm{W}}$ of eq. (15), but with the factor r in the denominator replaced by r^2 [69]. We call this the Ampére-Weber field, although it is the invention of the present authors.

2.1 Compatibility with Ampère's Force Law

We first show that the Ampère force law (2) for $d^2\mathbf{F}_{\mathrm{on}\ 1} = d^2\mathbf{F}_{\mathrm{A}}(\mathbf{x}_1, \mathbf{x}_2)$ on current element $I_1\, d\mathbf{l}_1$ at $\mathbf{x}_1$ can be related to the field $d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2)$ at $\mathbf{x}_1$ due to current element $I_2\, d\mathbf{l}_2$ at $\mathbf{x}_2$ by,

$$d^2\mathbf{F}_{\mathrm{A}}(\mathbf{x}_1, \mathbf{x}_2) = -I_1\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}\, (d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) + 2\boldsymbol{\nabla}[d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) \cdot \mathbf{r}]), \tag{20}$$

where,

$$d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mu_0}{4\pi} I_1\, d\mathbf{l}_2 \cdot \hat{\mathbf{r}}\, \frac{\hat{\mathbf{r}}}{r^2}\,. \tag{21}$$

From eq. (21), we have,

$$\begin{aligned} \boldsymbol{\nabla}[d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) \cdot \mathbf{r}] &= \frac{\mu_0}{4\pi} I_1 \boldsymbol{\nabla} \left(\frac{d\mathbf{l}_2 \cdot \mathbf{r}}{r^2} \right) \\ = \frac{\mu_0}{4\pi} I_1 \left(-\frac{2d\mathbf{l}_2 \cdot \mathbf{r}}{r^3} \boldsymbol{\nabla} r + \frac{\boldsymbol{\nabla}(d\mathbf{l}_2 \cdot \mathbf{r})}{r^2} \right) &= \frac{\mu_0}{4\pi} I_2 \left(-\frac{2d\mathbf{l}_2 \cdot \mathbf{r}}{r^4} \mathbf{r} + \frac{d\mathbf{l}_2}{r^2} \right) \\ &= \frac{\mu_0}{4\pi} I_2 \left(-\frac{2d\mathbf{l}_2 \cdot \hat{\mathbf{r}}}{r^2} \hat{\mathbf{r}} + \frac{d\mathbf{l}_2}{r^2} \right). \end{aligned} \tag{22}$$

Then,

$$\begin{aligned} &-I_1\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}\, (d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) + 2\boldsymbol{\nabla}[d\vec{\mathcal{B}}(\mathbf{x}_1, \mathbf{x}_2) \cdot \mathbf{r}]) \\ = -\frac{\mu_0}{4\pi} I_1\, d\mathbf{l}_1 \cdot &\left[I_2 (d\mathbf{l}_2 \cdot \hat{\mathbf{r}}) \frac{\hat{\mathbf{r}}}{r^2} + 2I_2 \left(-\frac{2d\mathbf{l}_2 \cdot \hat{\mathbf{r}}}{r^2} \hat{\mathbf{r}} + \frac{d\mathbf{l}_2}{r^2} \right) \right] \hat{\mathbf{r}} \\ &= \frac{\mu_0}{4\pi} I_1\, d\mathbf{l}_1 \cdot \left(3I_2 (d\mathbf{l}_2 \cdot \hat{\mathbf{r}}) \frac{\hat{\mathbf{r}}}{r^2} - \frac{2I_2\, d\mathbf{l}_2}{r^2} \right) \hat{\mathbf{r}} \\ &= \frac{\mu_0}{4\pi} I_1 I_2 \frac{3(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_2 \cdot \hat{\mathbf{r}}) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\, \hat{\mathbf{r}} \\ &= d^2\mathbf{F}_{\mathrm{A}}(\mathbf{x}_1, \mathbf{x}_2) = d^2\mathbf{F}_{\mathrm{on}\ 1}, \end{aligned} \tag{23}$$

in agreement with Ampère's form (2).

Thus, Ampère's force law (2) can be related to field $\vec{\mathcal{B}}$ if we allow the force law (20) to depend on the spatial derivatives of $\vec{\mathcal{B}}$ as well as $\vec{\mathcal{B}}$ itself. Such a derivative coupling is not favored in the simplest implementation of a field theory, but cannot be excluded altogether.

The total force on a current element $I_1\, d\mathbf{l}_1$ at position $\mathbf{x}_1$ due to circuit 2 is, according to Ampère's force law (23),

$$d\mathbf{F}_{\rm A}(\mathbf{x}_1) = \oint_2 d^2\mathbf{F}_{\rm A}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mu_0}{4\pi} I_1 I_2 \oint_2 \hat{\mathbf{r}}\, \frac{3(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_2 \cdot \hat{\mathbf{r}}) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2} . \tag{24}$$

As mentioned in sec. 1.1.1, Ampère's force law (24) for a current element $I_1\, d\mathbf{l}_1$ (or an electric charge q with velocity $\mathbf{v}$ where $q\,\mathbf{v} = I_1\, d\mathbf{l}_1$) differs from the Biot-Savart/Lorentz force, eq. (5), on the current element due to the magnetic field $\mathbf{B}$ at $\mathbf{x}_1$,

$$\begin{aligned} d\mathbf{F}_{\rm B-S}(\mathbf{x}_1) &= I_1\, d\mathbf{l}_1 \times \mathbf{B} = I_1\, d\mathbf{l}_1 \times \frac{\mu_0}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2} \\ &= \frac{\mu_0}{4\pi} I_1 I_2 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2}\, d\mathrm{Vol}_2, \end{aligned} \tag{25}$$

although the total force on a closed circuit due to another closed circuit is the same according to both Ampère's force law and the Biot-Savart/Lorentz force law. Hence, the Ampère-Weber magnetic field does not provide a good understanding of the forces between moving charges.

2.2 $\boldsymbol{\nabla} \cdot \vec{\mathcal{B}} = 0$

The divergence of $\vec{\mathcal{B}}(\mathbf{x})$ is, noting that $\boldsymbol{\nabla}$ acts on $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ but not on $\mathbf{J}(\mathbf{x}')$,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{\mathcal{B}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4}\, \mathbf{r} \right) d\mathrm{Vol}' \\ &= \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \boldsymbol{\nabla} \cdot \mathbf{r} + \mathbf{r} \cdot \boldsymbol{\nabla} \left(\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \right) \right] d\mathrm{Vol}' \\ &= \frac{\mu_0}{4\pi} \int \left[\frac{3\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} + \mathbf{r} \cdot \left(\frac{\boldsymbol{\nabla}(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})}{r^4} - \frac{4(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})}{r^5} \boldsymbol{\nabla} r \right) \right] d\mathrm{Vol}' \\ &= \frac{\mu_0}{4\pi} \int \left[\frac{3\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} + \mathbf{r} \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{r^4} - \frac{4(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})\mathbf{r}}{r^6} \right) \right] d\mathrm{Vol}' = 0, \end{aligned} \tag{26}$$

away from $r = 0$, *i.e.*, for source currents away from the observation point.

To ascertain the behavior of $\vec{\mathcal{B}}$ for small r, it is useful to consider its flux across the surface of a sphere of radius r, within which the current $\mathbf{J}$ is approximately constant.

$$\Phi = \int \vec{\mathcal{B}} \cdot \hat{\mathbf{r}}\, d\mathrm{Area} = \frac{\mu_0}{4\pi} \int_{-1}^{1} \frac{\mathbf{J} \cdot \hat{\mathbf{r}}}{r^2}\, 2\pi r^2\, d\cos\theta = \frac{\mu_0 J}{2} \int_{-1}^{1} \cos\theta\, d\cos\theta = 0, \tag{27}$$

taking the z-axis to be along the direction of $\mathbf{J}$ at the center of the sphere. That is, the magnetic field (19) for a current element $\mathbf{J}\, d\mathrm{Vol} = I\, d\mathbf{l}$ has lines of $\vec{\mathcal{B}}$ diverging from the current element in one hemisphere, and converging on it in the other, such that the total flux into/out of the current element is zero, as sketched in fig. 2. Then, together with eq. (26), we see that $\boldsymbol{\nabla} \cdot \vec{\mathcal{B}} = 0$ everywhere.

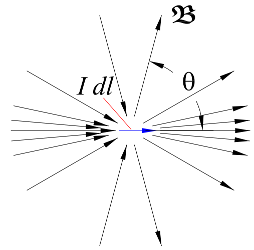

Fig. 2. Fields lines of $\vec{\mathcal{B}}$ for a current element $I\, d\mathbf{l}$.

2.3 A Vector Potential for $\vec{\mathcal{B}}$

Since $\boldsymbol{\nabla} \cdot \vec{\mathcal{B}} = 0$ everywhere, there exists a vector potential $\vec{\mathcal{A}}$ such that $\vec{\mathcal{B}} = \boldsymbol{\nabla} \times \vec{\mathcal{A}}$. A particular form of the vector potential is,

$$\vec{\mathcal{A}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{2r^2}\, d\mathrm{Vol}', \tag{28}$$

noting that,

$$\begin{aligned} \boldsymbol{\nabla} \times \left(\frac{\mathbf{J} \times \mathbf{r}}{r^2} \right) &= \frac{\mathbf{J}}{r^2}(\boldsymbol{\nabla} \cdot \mathbf{r}) - \mathbf{r} \left(\boldsymbol{\nabla} \cdot \frac{\mathbf{J}}{r^2} \right) + (\mathbf{r} \cdot \boldsymbol{\nabla}) \frac{\mathbf{J}}{r^2} - \left(\frac{\mathbf{J}}{r^2} \cdot \boldsymbol{\nabla} \right) \mathbf{r} \\ &= \frac{3\mathbf{J}}{r^2} - \mathbf{r} \left(\mathbf{J} \cdot \boldsymbol{\nabla} \frac{1}{r^2} \right) - \frac{2\mathbf{J}}{r^3} (\mathbf{r} \cdot \boldsymbol{\nabla}) r - \frac{\mathbf{J}}{r^2} \\ &= \frac{2\mathbf{J}}{r^2} + 2\mathbf{r} \frac{\mathbf{J} \cdot \mathbf{r}}{r^4} - \frac{2\mathbf{J}}{r^2} = \frac{2(\mathbf{J} \cdot \mathbf{r})\, \mathbf{r}}{r^4} \propto d\vec{\mathcal{B}}. \end{aligned} \tag{29}$$

2.4 $\boldsymbol{\nabla} \times \vec{\mathcal{B}}$

As noted by Helmholtz, Theorem VI, p. 61 of [70] (1858), to specify a vector field via first-order differential equations, both the curl and the divergence of the field must be known [71]. For the usual magnetic field $\mathbf{B}$, its curl for steady-state examples is,

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{30}$$

which is often called “Ampère’s Law” (sec. 1.1.2).

The curl of $\vec{\mathcal{B}}$ is,

$$
\begin{aligned}
\nabla \times \vec{\mathcal{B}}(\mathbf{x}) &= \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \mathbf{r} \right) d\mathrm{Vol}' \\
&= \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \nabla \times \mathbf{r} - \mathbf{r} \times \nabla \left(\frac{\mathbf{J}(\mathbf{x}') \cdot \mathbf{r}}{r^4} \right) \right] d\mathrm{Vol}' \\
&= -\frac{\mu_0}{4\pi} \int \mathbf{r} \times \left(\frac{\nabla (\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})}{r^4} - \frac{4(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})}{r^5} \nabla r \right) d\mathrm{Vol}' \\
&= -\frac{\mu_0}{4\pi} \int \mathbf{r} \times \left(\frac{\mathbf{J}(\mathbf{x}')}{r^4} - \frac{4(\mathbf{J}(\mathbf{x}') \cdot \mathbf{r})\, \mathbf{r}}{r^6} \right) d\mathrm{Vol}' \\
&= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^4} \, d\mathrm{Vol}'. \qquad (31)
\end{aligned}
$$

This is nonzero throughout all space, and does not lend itself to a simple physical interpretation as to the source of the magnetic field, in contrast to Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, for the usual (static) magnetic field $\mathbf{B}$.

2.5 Three Examples

2.5.1 Magnetic Dipole

We now consider a magnetic dipole $\mathbf{m} = \pi a^2 I\, \hat{\mathbf{z}}$, *i.e.*, a small, circular loop of radius a, centered on the origin, that carries steady current I, as sketched in fig. 3.

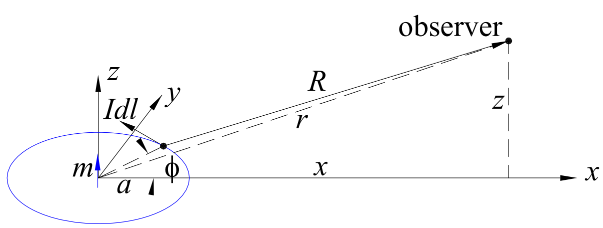

Fig. 3. Geometry for an observer in the x-z plane of a current loop of radius a, in the x-y plane about the origin.

We calculate $\vec{\mathcal{B}}$ at the point $\mathbf{r} = (x \gg a, 0, z)$, with $r = \sqrt{x^2 + z^2}$. For a current element $I\, dl = I a\, d\phi$ located at angle ϕ to the z-axis, *i.e.*, at $\mathbf{r}' = (a \cos\phi, 0, a \sin\phi)$ in (x, y, z) coordinates, we have, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$,

$$
\begin{align}
d\mathbf{l} &= a\, d\phi\, (-\sin\phi, 0, \cos\phi), \tag{32}\\
\mathbf{R} &= (x - a\cos\phi, -a\sin\phi, z), \tag{33}\\
d\mathbf{l}\cdot\mathbf{R} &= -ax\, d\phi \sin\phi, \tag{34}
\end{align}
$$

$$
R = \sqrt{x^2 - 2ax\cos\phi + a^2 + z^2} \approx \sqrt{x^2+z^2}\left(1 - \frac{ax\cos\phi}{x^2+z^2}\right) = r\left(1 - \frac{ax\cos\phi}{r^2}\right), \tag{35}
$$

$$
\begin{align}
\mathbf{B}_{\text{A−W}} &= \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} \tag{36}\\
&\approx \frac{\mu_0}{4\pi}\frac{aI}{r^4}\int_0^{2\pi} d\phi\,(-x\sin\phi)\left(1 + \frac{4ax\cos\phi}{r^2}\right)(x - a\cos\phi, -a\sin\phi, z)\\
&= \frac{\mu_0}{4\pi}\frac{aI}{r^4}(0, \pi ax, 0) = \frac{\mu_0}{4\pi}\frac{mx}{r^4}\,\hat{\mathbf{y}} = \frac{\mu_0}{4\pi}\frac{mx}{r^4}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^4} = \boldsymbol{\nabla}\times\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{2r^2} = \boldsymbol{\nabla}\times\mathbf{A}_{\text{A−W}}.
\end{align}
$$

Lines of the $\vec{\mathcal{B}} \propto \mathbf{m}\times\hat{\mathbf{r}}/r^3$ are circles centered on the axis $\mathbf{m}$, as sketched in fig. 4, and do not at all resemble the pattern of iron filings found by Faraday for a small dipole magnet (Fig. 1).

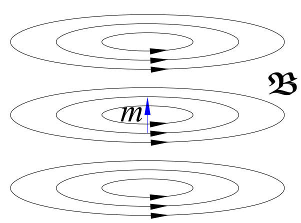

Fig. 4. Field lines of $\vec{\mathcal{B}}$ for a current loop about the origin in the x-y plane. The $1/r^3$ dependence of $\vec{\mathcal{B}}$ is not well represented in the figure.

Hence, while the field $\vec{\mathcal{B}}$ is mathematically consistent with Ampère's force law (2) between two circuits with steady currents, it seems unappealing physically, and would not have been accepted by Faraday.

2.5.2 Infinite Solenoid

In this section, we consider an infinite solenoid of radius a along the z-axis, with steady, azimuthal surface current I per unit length in z, as sketched in fig. 5.

We calculate $\vec{\mathcal{B}}$ at the point $\mathbf{r} = (r, 0, 0)$. For a current element $I\,dl = Ia\,d\phi$ located at angle

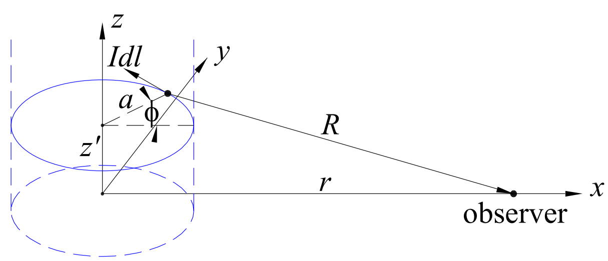

Fig. 5. Geometry for an observer on the x-axis outside an infinite solenoid of radius a along the z-axis.

ϕ to the x-axis at height z', *i.e.*, at $\mathbf{r}' = (a\cos\phi, a\sin\phi, z')$, we have, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$,

$$d\mathbf{l} = a\,d\phi\,(-\sin\phi, \cos\phi, 0), \tag{37}$$

$$\mathbf{R} = (r - a\cos\phi, -a\sin\phi, -z'), \tag{38}$$

$$d\mathbf{l}\cdot\mathbf{R} = -ar\,d\phi\sin\phi, \tag{39}$$

$$R = \sqrt{r^2 - 2ar\cos\phi + a^2 + z'^2}, \tag{40}$$

$$\begin{aligned}\vec{\mathcal{B}} = \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} &= -\frac{\mu_0}{4\pi}arI\int_{-\infty}^{\infty}dz'\int_0^{2\pi}d\phi\,\frac{\sin\phi\,(r - a\cos\phi, -a\sin\phi, -z')}{(r^2 - 2ar\cos\phi + a^2 + z'^2)^2}\\ &= \frac{\mu_0}{4\pi}\frac{\pi arI}{2}\int_0^{2\pi}d\phi\,\frac{(\sin\phi(a\cos\phi - r), a\sin^2\phi, 0)}{(r^2 - 2ar\cos\phi + a^2)^{3/2}},\end{aligned} \tag{41}$$

using Dwight 120.2 [73]. This is a nonzero function for any value of the distance r of the observer from the axis of the infinite solenoid. The x-component of the final integral is zero, leaving only the y-component, which is also in the $\hat{\boldsymbol{\phi}}$-direction at the observer. The character of $\vec{\mathcal{B}}$ is in contrast to the Biot-Savart magnetic field (6) which is zero outside the solenoid and constant inside (with value $\mu_0 I\,\hat{\mathbf{z}}$). For $r \gg a$ (outside the solenoid),

$$\vec{\mathcal{B}}(r \gg a) \to \frac{\mu_0}{4\pi}\frac{\pi^2 a^2 rI}{2r^3}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0}{4\pi}\frac{\pi\mathbf{m}\times\mathbf{r}}{2r^3}, \tag{42}$$

where $\mathbf{r} = (r, 0, 0)$ and $\mathbf{m} = \pi a^2 I\,\hat{\mathbf{z}}$ is the magnetic dipole moment per unit length of the infinite solenoid. That is, the field lines of $\vec{\mathcal{B}}$ for an infinite solenoid are of the same circular form as those for a magnetic dipole (sec. 2.5.1 above).

2.5.3 Long, Straight Wire

In this section, we consider a wire along the z-axis, carrying current I.

The Ampère-Weber field (19) at the point $\mathbf{r} = (x, y, z) = (r, 0, 0)$ is, integrating over current elements $I\,d\mathbf{l}$ at $\mathbf{r}' = (0, 0, z')$,

$$d\mathbf{l} = (0, 0, dz'), \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}' = (r, 0, -z'), \tag{43}$$

$$d\mathbf{l}\cdot\mathbf{R} = -z'\,dz', \qquad R = \sqrt{r^2 + z'^2}, \tag{44}$$

$$\vec{\mathcal{B}} = \frac{\mu_0}{4\pi}\oint I\frac{(d\mathbf{l}\cdot\mathbf{R})\mathbf{R}}{R^4} = -\frac{\mu_0}{4\pi}I\int_{-\infty}^{\infty}dz'\,\frac{z'(r, 0, -z')}{(r^2 + z'^2)^2} = \frac{\mu_0 I}{8r}\,\hat{\mathbf{z}}, \tag{45}$$

using Dwight 122.2 [73]. That is, $\vec{\mathcal{B}}$ is parallel to the wire and falls off inversely with the distance from it.

This result contrasts with Faraday's vision that the lines of magnetic field $\mathbf{B}$ circle about long, straight, current-carrying wires (Art. 233 of [74]; see also sec. A.17.4 of [4]).

An important insight of Ampère was that all magnetism is due to electric currents, rather than to magnetic poles as had been assumed by all previous workers. In particular (as mentioned in [28]), Biot and Savart studied the interaction of a magnetic needle with an electric current, supposing that a magnetic pole p resided on the tip of the needle, such that the force law they proposed can be written (in vector form) as,

$$\mathbf{F} = p\,\mathbf{B}_{\mathrm{B-S}}, \quad \text{where} \quad \mathbf{B}_{\mathrm{B-S}} = \frac{\mu_0}{4\pi}\oint \frac{I\,d\mathbf{l}\times\hat{\mathbf{r}}}{r^2}\,. \tag{46}$$

For the case of steady current $I\,\hat{\mathbf{z}}$ in a long straight wire, the Biot-Savart magnetic field is $\mathbf{B}_{\mathrm{B-S}} = \mu_0 I\,\hat{\mathbf{z}}\times\hat{\mathbf{r}}/2\pi r$ at the magnetic pole p at (transverse) distance $\mathbf{r}$ from the wire.

If we consider that an alternative magnetic field must also describe the force on a magnetic pole according to $\mathbf{F} = p\,\vec{\mathcal{B}}_{\mathrm{alt}}$, then it is clear that the form $\vec{\mathcal{B}}$ of eq. (19) does not satisfy this. In particular, for the case of the magnetic field (45) due to the current in a long, straight wire (as in the experiment of Biot and Savart [29]), $\vec{\mathcal{B}} = \mu_0 I\,\hat{\mathbf{z}}/8r$, eq. (45), is parallel to the wire, which would imply a force on the pole parallel to the wire, rather than in the direction $\mathbf{I}\times\mathbf{r}$ as observed experimentally.

3 Summary

We have shown that it is possible to relate Ampère's differential-force law (2) to a vector field, that we call the Ampère-Weber field, $\vec{\mathcal{B}}$, eq. (19), which is very different from the usual magnetic field $\mathbf{B}$ of Maxwellian electrodynamics. However, the Ampère-Weber field has many conceptual defects, and does not lead to a full theory of electrodynamics. Rather, it is mainly a mathematical curiosity. It is perhaps just as well that the Ampère-Weber field was not invented in the 1800's.

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English translation in [2]. A thoughtful online site about Ampère is [3]. For a historical survey of the development of electrodynamics in the 1800's, see, for example, the Appendix of [4].

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kirkmcd.princeton.edu/examples/EM/assis_ampere_15.pdf
Includes an English translation of [1].

[3] *Ampère et l'histoire de l'électricité*, http://www.ampere.cnrs.fr/parcourspedagogique/index-en.php

[4] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), kirkmcd.princeton.edu/examples/faradaydisk.pdf

[5] Ampère sometimes used the notation that the angle between $d\mathbf{l}$ and $\mathbf{r}$ is θ, and the angle between the plane of $d\mathbf{l}$ and $\mathbf{r}$ and that of $d\mathbf{l}'$ and $\mathbf{r}$ is ω. Then, $d\mathbf{l}\cdot d\mathbf{l}' = dl\, dl'(\sin\theta\sin\theta'\cos\omega+\cos\theta\cos\theta')$, and the force element of eq. (2) can be written as,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I I'\, dl\, dl(\cos\theta\cos\theta' - 2\sin\theta\sin\theta'\cos\omega)\frac{\hat{\mathbf{r}}}{r^2} \tag{47}$$

Ampère also noted the equivalents to,

$$d\mathbf{l} = \frac{\partial\mathbf{r}}{\partial l}\, dl, \tag{48}$$

$$\mathbf{r}\cdot d\mathbf{l} = \mathbf{r}\cdot\frac{\partial\mathbf{r}}{\partial l}\, dl = \frac{1}{2}\frac{\partial r^2}{\partial l}\, dl = r\frac{\partial r}{\partial l}\, dl, \tag{49}$$

$$d\mathbf{l}' = -\frac{\partial\mathbf{r}}{\partial l'}\, dl', \qquad \mathbf{r}\cdot d\mathbf{l}' = -r\frac{\partial r}{\partial l'}\, dl', \tag{50}$$

where l and l' measure distance along the corresponding circuits in the directions of their currents. Then,

$$\begin{aligned} d\mathbf{l}\cdot d\mathbf{l}' &= -d\mathbf{l}\cdot\frac{\partial\mathbf{r}}{\partial l'}\, dl' = -\frac{\partial}{\partial l'}(\mathbf{r}\cdot d\mathbf{l})\, dl' = -\frac{\partial}{\partial l}\left(r\frac{\partial r}{\partial l}\right) dl\, dl' \\ &= -\left(\frac{\partial r}{\partial l}\frac{\partial r}{\partial l'} + r\frac{\partial^2 r}{\partial l'\partial l'}\right) dl\, dl', \end{aligned} \tag{51}$$

and eq. (2) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I I'\, dl\, dl'\left[2r\frac{\partial^2 r}{\partial l\partial l'} - \frac{\partial r}{\partial l}\frac{\partial r}{\partial l'}\right]\frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I I'\, dl\, dl'\frac{\partial^2\sqrt{r}}{\partial l\partial l}\frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\text{on }2}. \tag{52}$$

See pp. 358-361 and 380-381 of [2].

[6] The permeability μ, where $\mathbf{B} = \mu\mathbf{H}$ for a linear magnetic medium, was called the magnetic inductive capacity by Maxwell in Art. 627 of [7]. The permeability of the vacuum began to be called μ_0 in the 1890's, perhaps beginning with Lodge, item 14 of [8], but its value was subject to the "4π" debate initated by Heaviside [9]. See, for example, p. 104 of [10]. μ_0 became an "official" contstant of Nature upon the adoption of the MKS system of units in 1933 [11].

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[12] Maxwell called Ampère the "Newton of electricity" in Art. 528 of [7].

[13] In Arts. 525-527 of his *Treatise* [7], Maxwell showed that Ampère's force law (1) gave the same result for four different expressions for $d^2\mathbf{F}_\mathrm{A}$, all functions of $I\,d\mathbf{l}$, $I'\,d\mathbf{l}'$ and $\mathbf{r}$ [14], concluding that Ampère's form, eq. (42) of Art. 526 and our eq. (2), "is undoubtedly the best". Maxwell's eq. (43) of Art. 526, which he attributed to Grassmann [15], corresponds to our eq. (4). Apparently, Maxwell did not realize that his vector-diffential equations for the magnetic field lead most directly to the Biot-Savart-Grassmann form.

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[28] Biot and Savart [29,30] actually studied only the force due to an electric current I in a wire on one pole, p, of a long, thin magnet. Their initial interpretation of the results was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [31,32] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F} = \frac{\mu_0 \, p}{4\pi} \oint \frac{I \, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \,, \tag{53}$$

where $\mathbf{r}$ is the distance from a current element $I \, d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (53) in terms of a magnetic field, $\mathbf{B} = \mathbf{F}/p$.

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[31] J.-B. Biot, *Sur l'Aimantation imprimée aux métaux par l'électricité en mouvement*, J. Savans **11**, 221-235 (1821).
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[32] J.-B. Biot, *Précis Élèmentaire de Physique Expérimentale*, 3rd ed., Vol. 2 (Paris, 1820), pp. 704-774.
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http://kirkmcd.princeton.edu/examples/EM/abraham_foppl_elektrizitat_v1_04.pdf

[38] J.C. Maxwell, *On Physical Lines of Force. Part I.—The Theory of Molecular Vortices applied to Magnetic Phenomena*, Phil. Mag. **21**, 161-175 (1861).
https://doi.org/10.1080/14786446108643033
http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_161_61.pdf

[39] Faraday first mentioned *magnetic lines of force* in Art. 114 of [40] (1831): *By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.*

In 1845, Art. 2247 of [41], the term *magnetic field* appears for the first time in print: *The ends of these bars form the opposite poles of contrary name; the magnetic field between them can be made of greater or smaller extent, and the intensity of the lines of magnetic force be proportionately varied.*

In 1852, Faraday published a set of speculative comments [42] in the Phil. Mag. (rather than Phil. Trans. Roy. Soc. London, the usual venue for his *Experimental Researches*), arguing more strongly for the physical reality of the lines of force.

In Art. 3258 0f [42] he considered the effect of a magnet in vacuum, concluding (perhaps for the first time) that the lines of force have existence independent of a material medium:

A magnet placed in the middle of the best vacuum we can produce...acts as well upon a needle as if it were surrounded by air, water or glass; and therefore these lines exist in such a vacuum as well as where there is matter.

[40] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125-162 (1832). doi.org/10.1098/rstl.1832.0006
kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_125_32.pdf

[41] M. Faraday, *Experimental Researches in Electricity.—Twentieth Series*, Phil. Trans. Roy. Soc. London **136**, 21-40 (1846). doi.org/10.1098/rstl.1846.0003
kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_136_21_46.pdf

[42] M. Faraday, *On the Physical Character of the Lines of Magnetic Force*, Phil. Mag. **3**, 401-428 (1852). https://doi.org/10.1080/14786445208647033
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https://www.google.com/books/edition/The_Cambridge_and_Dublin_Mathematical_Jo/TxcAAAAAMAAJ?hl=en
kirkmcd.princeton.edu/examples/EM/thomson_cdmj_2_61_47.pdf

[45] In 1867 Gauss posthumously published an analysis that he dated to 1835 (p. 609 of [46]), in which he stated that a time-dependent electric current leads to an electric field which is the time derivative of what we now called the vector potential. English translation from [47]:

The Law of Induction

Found out Jan. 23, 1835

at 7 a.m. before getting up.

The electricity producing power, which is caused in a point P by a current-element $\boldsymbol{\gamma}$, at a distance from P, $= r$, is during the time dt the difference in the values of $\boldsymbol{\gamma}/r$ corresponding to the moments t and dt, divided by dt. where $\boldsymbol{\gamma}$ is considered both with respect to size and direction. This can be expressed briefly and clearly by

$$-\frac{\mathrm{d}(\boldsymbol{\gamma}/r)}{\mathrm{d}t}\,. \tag{54}$$

Gauss' unpublished insight that electromagnetic induction is related to the negative time derivative of a scalar quantity was probably communicated in the late 1830's to his German colleagues, of whom Weber was the closest.

On p. 612 of [46] (presumably also from 1835), Gauss noted a relation (here transcribed into vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential Ω of a circuit with unit electrical current (which he related to the solid angle subtended by the circuit on p. 611), $-\boldsymbol{\nabla}\Omega = \boldsymbol{\nabla}\times\mathbf{A}$. While we would identify $-\boldsymbol{\nabla}\Omega$ with the magnetic field $\mathbf{H}$, Gauss called it the "electricity-generating force". In any case, this is the earliest (claimed) appearance of the curl operator (although published later than MacCullagh's use of it, p. 22 of [48] (1839).

[46] C.F. Gauss, *Werke*, Vol. 5 (Göttingen, 1867).
https://www.google.com/books/edition/Carl_Friedrich_Gauss_Werke/HwgTAQAAMAAJ?hl=en&gbpv=1&printsec=frontcover
kirkmcd.princeton.edu/examples/EM/gauss_werke_v5.pdf

[47] J.J. Roche, *A Critical Study of the Vector potential*, in *Physicists Look Back: Studies in the History of Physics*, J.J. Roche ed. (Adam Hilger, 1990), p. 144.
kirkmcd.princeton.edu/examples/EM/roche_144_90.pdf

[48] J. MacCullagh, *An Essay toward a dynamical Theory of crystalline Reflexion and Refraction*, Trans. Roy. Irish Acad. **21**, 17-50 (1839). https://www.jstor.org/stable/30078974
kirkmcd.princeton.edu/examples/optics/maccullagh_tria_21_17_39.pdf

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Abh. König. Sächs. Gesell. Wiss. 209-378 (1846).
https://books.google.com/books/about/Elektrodynamische_Maassbestimmungen.html?hl=de&id=yE1R6pUZjFMC
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English translation, p. 33 of [50].

[50] A.K.T. Assis, ed., *Wilhelm Weber's Main Works on Electrodynamics Translated into English*, Vol. II (Apeiron, 2021).
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http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_281_61.pdf
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[52] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459-512 (1865). doi/10.1098/rstl.1865.0008
kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

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http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_102_529_57.pdf
English translation in P. Graneau and A.K.T. Assis, *Kirchhoff on the Motion of Electricity in Conductors*, Apeiron **19**, 19-25 (1994).
https://www.ifi.unicamp.br/~assis/Apeiron-V19-p19-25(1994).pdf
http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_102_529_57_english.pdf

[54] W. Weber and R. Kohlrausch, *Ueber die Elektricitätsmenge, welche bei galvanischen Strömen durch den Querschnitt der Kette fliesst*, Ann. d. Phys. **99**, 10 (1856).
https://onlinelibrary.wiley.com/doi/10.1002/andp.18561750903
kirkmcd.princeton.edu/examples/EM/weber_ap_99_10_56.pdf
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English translation, p. 131 of [55].

[55] A.K.T. Assis, ed., *Wilhelm Weber's Main Works on Electrodynamics Translated into English*, Vol. III (Apeiron, 2021).
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[56] W. Weber, *Elektrodynamische Maassbestimmungen*, Abh. König. Sächs. Gesell. Wiss. **6**, 573-718 (1864),
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http://kirkmcd.princeton.edu/examples/EM/weber_aksgw_6_571_64.pdf
English translation,p. 131 of [55].

[57] H, Helmholtz, *Ueber die Bewenungsgleichungen der Elektricität für ruhende leitende Körper*, J. Reine Angew. Math. **72**, 57-129 (1870). https://doi.org/10.1515/crll.1870.72.57
http://kirkmcd.princeton.edu/examples/EM/helmholtz_jram_72_57_70.pdf

[58] Commentaries on Helmholtz' paper [57] are given in [59]- [63].

[59] A.E. Woodruff, *The Contributions of Hermann von Helmholtz to Electrodynamics*, Isis **59**, 300-311 (1968). http://www.jstor.org/stable/227884
http://kirkmcd.princeton.edu/examples/EM/woodruff_isis_59_300_68.pdf

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[62] J. Buchwald, *The Creation of Scientific Effects* (U. Chicago Press, 1994), Chap. 6.

[63] K.T. McDonald, *Helmholtz and the Velocity Gauge* (Mar. 31, 2018),
http://kirkmcd.princeton.edu/examples/velocity.pdf

[64] F.E. Neumann, *Über ein allgemeines Princip der mathematischen Theorie inducirter elektrischer Ströme*, Abh. König. Akad. Wiss. Berlin, pp. 1-71 (1847).
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English translation on pp. 207-248 of
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[66] See also sec. IIB of [34], and [14]. The energy that Helmholtz associated with Weber was never actually advocated by the latter, who had a different vision of magnetic energy, as reviewed in sec. A.23 of [4].

[67] Thus, we cannot write for an isolated current element that
$d\mathbf{A} = \mu_0 I[(1+k)d\mathbf{l} + (1-k)(d\mathbf{l}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}]/8\pi r$.

[68] Helmholtz' discussion was tacitly restricted to electro- and magnetostatics, such that his eq. (3^a), p. 80 of [57], $\boldsymbol{\nabla}\cdot\mathbf{A} = k\,dV/dt$, where V is the instantaneous electric scalar potential,

led him to identify $k = 0$ with Maxwell's theory [52] which emphasized $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, although we would now consider $k = 1$ to be compatible with Maxwell's theory for static electromagnetism. Maxwell was more interested in electrodynamics than electro/magnetostatics, such that his only mention of the "Neumann" magnetostatic scalar potential, our eq. (9), was in his eq. (9), Art. 422 of [7].

[69] A reviewer noted that $\vec{\mathcal{B}}$, eq. (19), is a vector field while the usual form for $\mathbf{B}$, eq. (6), is a pseudovector field.

[70] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, Crelles J. **55**, 25-55 (1858).
http://kirkmcd.princeton.edu/examples/fluids/helmholtz_jram_55_25_58.pdf
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On Integrals of the Hydrodynamical Equations, which express Vortex-motion, Phil. Mag. **33**, 485-512 (1867). https://doi.org/10.1080/14786446708639824
http://kirkmcd.princeton.edu/examples/fluids/helmholtz_pm_33_485_67.pdf

[71] The "Helmholtz decompotion" was first demonstrated on pp. 9-10 of G.G. Stokes, *On the Dynamical Theory of Diffraction*, Trans. Camb. Phil. Soc. **9**(1), 1-62 (1849).
http://kirkmcd.princeton.edu/examples/optics/stokes_tcps_9(1)_1_51.pdf
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See also [72].

[72] K.T. McDonald, *On the "Helmholtz" Decomposition* (Apr. 23, 2023),
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[73] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4$^{\text{th}}$ ed. (Macmillan, 1961).
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