Is There a Maximum Z for an Atom?

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(June 2, 2010)

1 Problem

Is there a maximum electric charge Ze for the nucleus of a stable atom, where $-e$ is the charge of an electron?

2 Solution

According to nonrelativistic quantum mechanics there is no obvious maximum Z for the nucleus of an atom. However, the binding energy of a K-electron,

$$E = \frac{(Z\alpha)^2 mc^2}{2}, \tag{1}$$

where $\alpha = e^2/\hbar c \approx 1/137$, equals the rest energy mc^2 of an electron when $Z = \sqrt{2}/\alpha \approx 194$, which suggests that relativistic corrections may be prominent for large Z. And indeed, the relativistic variant of the Bohr atom [1] given by Sommerfeld [2] predicted the binding energy of a K-electron to be

$$E = \left(1 - \frac{1}{\sqrt{1 + (Z\alpha)^2 / \left[n_r + \sqrt{n_\phi^2 - (Z\alpha)^2}\right]^2}}\right) mc^2, \tag{2}$$

with $n_r = 0$ and $n_\phi = 1$, which expression approaches mc^2 as Z approaches $1/\alpha$ and is complex for larger Z. An identical expression is obtained for a Dirac electron in the Coulomb potential of a point nucleus of charge Ze [3, 4], although the interpretation of the quantum numbers n_r and n_ϕ is different ($n_\phi = j + 1/2$, $n_r = n - j - 1/2$).

The question of atoms with large Z seems to have been little considered in the "heroic" era of quantum theory, with the first, brief mention of this topic perhaps given by Schiff *et al.* [5] in 1939. They note that for a bare nucleus of large Z the lowest energy state is not simply nucleus + vacuum. It appears that not until 1968 was it noted (by Pieper and Greiner [6]) that the lowest-energy state for a point nucleus with $Z\alpha > 1$ consists of a one-electron atom + free positron. For a real nucleus with finite radius the critical value of Z, beyond which atoms are always accompanied by positrons, is somewhat larger than $1/\alpha$, as reviewed in [7].

Independent of the issue of whether nuclei can exist with $Z\alpha \gtrsim 1$, atoms with such nuclei could not be regarded as stable, in that they could interact with their accompanying positron(s) and change their state (at least temporarily). However, since there are no long-lived nuclei with $Z \gtrsim 100$, this question is "academic".

In principle, a short-lived, high-Z state can be produced in the collision of heavy nuclei, such that one or more positrons might be emitted from the collision region, which latter then contains an equal number of electrons. Despite some early controversial experimental results, it is considered that no such effect has been observed experimentally to date [8].

A version of this argument in which one of the electron or positron becomes bound into a gravitational "orbit" underlies the concept of Hawking radiation [9, 10].

References

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